Turbulence without vortex stretching

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Introduction

▪ Turbulence without vortex stretching (VS)

- Consider the curl of Navier-Stokes equation: vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \underbrace{(\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}}_{\text{Advection}} - \nu \Delta \boldsymbol{\omega} = \underbrace{(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}}_{\text{Vortex stretching}}$$

(3D, incompressible, statistically homogeneous, force-free)

($\boldsymbol{u}$: velocity, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$: vorticity, ν: viscosity)

- Removing VS term is equivalent to applying an artificial forcing term

$$\frac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla P - \boldsymbol{f}, \text{ where } \nabla \times \boldsymbol{f} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$$

Bos, Wouter JT. "Three-dimensional turbulence without vortex stretching." *Journal of Fluid Mechanics* 915 (2021).

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Introduction

▪ Turbulence without vortex stretching (VS)

➢ Consider the curl of Navier-Stokes equation: vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \underbrace{(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}}_{\text{Advection}} - \nu\Delta\boldsymbol{\omega} = \underbrace{\cancel{(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}}}_{\text{Vortex stretching}} = \mathbf{0}$$

(3D, incompressible, statistically homogeneous, force-free)

($\boldsymbol{u}$: velocity, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$: vorticity, ν: viscosity)

➢ Removing VS term is equivalent to applying an artificial forcing term

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▪ Objective

- Study the vortex stretching by eliminating it

  VS plays an important role in the transfer of kinetic energy, affects the rate of dissipation.

- Bridge of 2-dimensional (2D) and 3-dimensional (3D) turbulence

  2D turbulence: $\boldsymbol{\omega} \perp \boldsymbol{u} \;\; \Rightarrow \;\; (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = 0$

  Study 3D turbulence containing 2D properties

▪ Two parts of the presentation

Inviscid, truncated: Reach an absolute equilibrium state

Viscous: A dual cascade

Inviscid turbulence

Absolute equilibrium

- **Normal 3D turbulence**

➢ A truncated, inviscid, force-free system will finally reach a steady state:

Absolute equilibrium state

➢ Statistical mechanics:

Conservation of energy and helicity (Kraichnan 1973, Lee 1952)

Mean velocity per unit volume: $U(k) = 2\alpha/(\alpha^2 - \beta^2 k^2)$

➢ Low helicity state ($\beta \approx 0$):

$U(k) \approx cst$: all Fourier modes contain statistically the same amount of energy (**equipartition of energy**)

$$E(k) \sim 4\pi k^2 U(k) \propto k^2$$

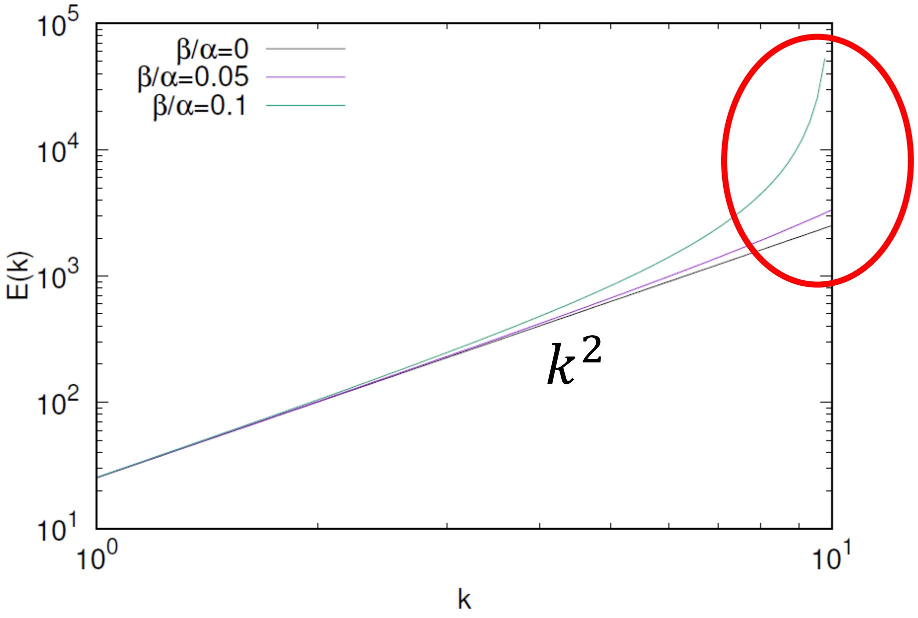
Kraichnan, Robert H. "Helical turbulence and absolute equilibrium." *Journal of Fluid Mechanics* 59.4 (1973): 745-752.

Lee, T. D. "On some statistical properties of hydrodynamical and magneto-hydrodynamical fields." *Quarterly of Applied Mathematics* 10.1 (1952): 69-74.

Absolute equilibrium

- Helical turbulence ($\beta \neq 0$):

  Helicity affects only small scales (damped by viscous dissipation)



$$E(k) = \frac{8\pi\alpha k^2}{\alpha^2 - \beta^2 k^2}$$

(Kraichnan 1973)

- **3D turbulence without VS?**
  - Inviscid invariants
  - Energy spectrum

Numerical approach

- **Simulation method**

  DNS, Periodic box in three dimensions, Grid $128 \times 128 \times 128$

- **Governing equation of velocity**

  Take the curl of the vorticity equation $\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = \mathbf{0}$

  Transform into Fourier space

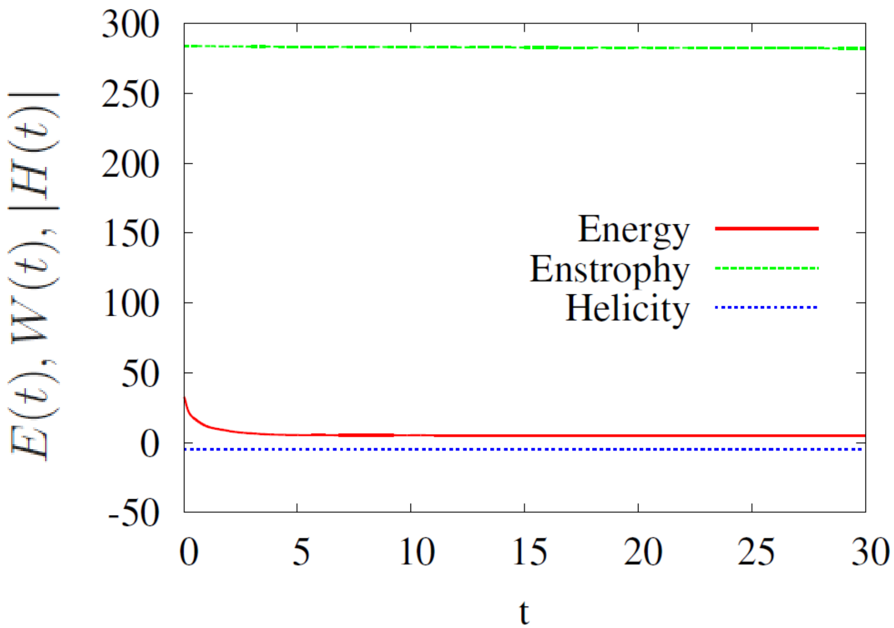
$$\frac{\partial \hat{\boldsymbol{u}}}{\partial t} = -\frac{i}{k^2}\boldsymbol{k} \times \underbrace{\mathcal{F}[(\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}]}_{\text{pseudo-spectral}}$$

($\hat{\boldsymbol{u}}$: velocity vector in Fourier space)

Invariants: Helicity & Enstrophy

- **Time evolution of statistics**

Random initial velocity field



- Helicity: $\frac{dH}{dt} = 0$
- Enstrophy: $\frac{dW}{dt} = 0$
- Energy: $\frac{dE}{dt} \neq 0$

- **2D & 3D**

2D turbulence: energy and enstrophy, 3D turbulence: energy and helicity

3D no vortex stretching turbulence: **between 2D and 3D**

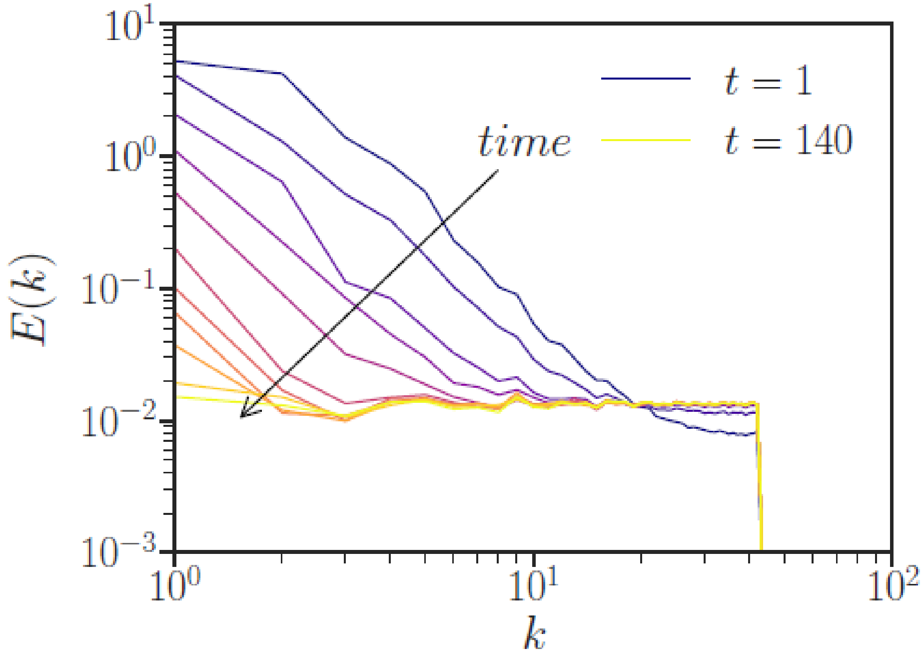
▪ Absolute equilibrium without VS (helicity-free)

➢ Hypothesis: equipartition of **enstrophy** for **helicity-free** turbulence

Enstrophy spectrum: $W(k) \propto k^2$

Energy spectrum: $\boxed{E(k)} = k^{-2}W(k) \propto \boxed{k^0}$

➢ Numerical verification:
Helicity-free initial field



▪ Absolute equilibrium without VS (helical turbulence)

- Theoretical analysis:

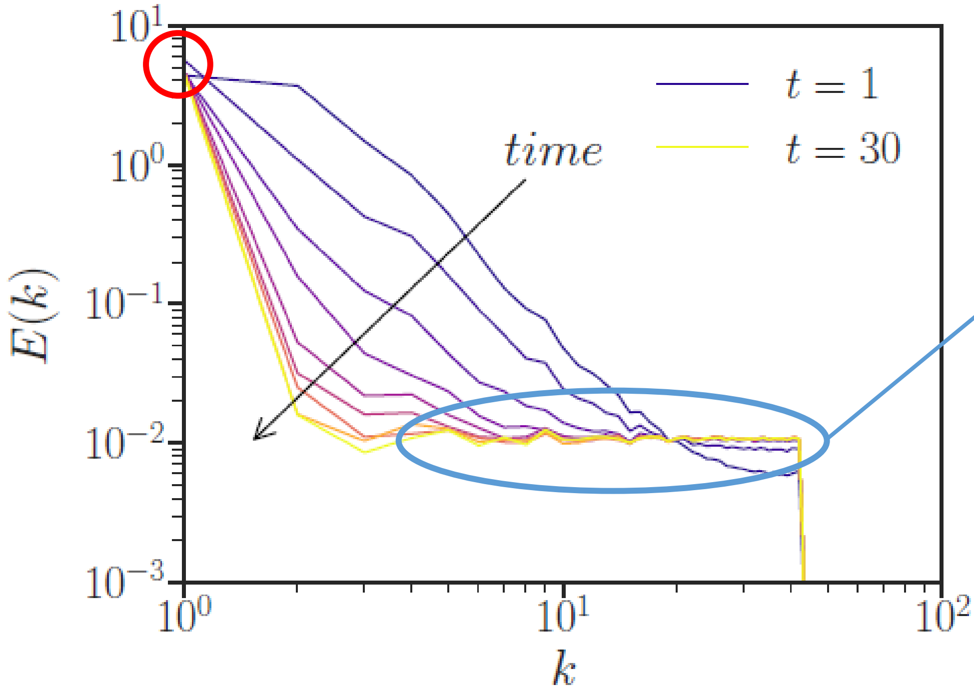
  Statistical mechanics (Kraichnan 1973)

$$E(k) = \frac{8\pi}{\alpha}\frac{k^2}{k^2-\gamma^2} \qquad (\gamma = \beta/\alpha)$$

  α, γ: determined by enstrophy and helicity (helicity-free: $\gamma = 0$)

- Numerical simulation:

  Helical, random initial field

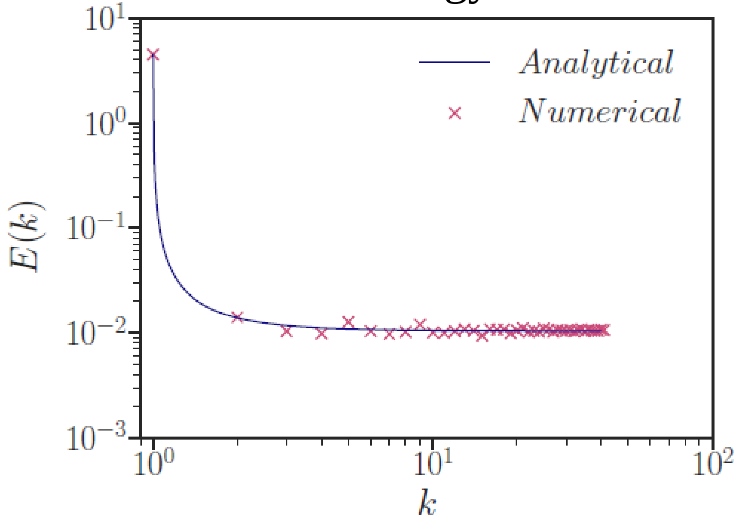


$E(k) \propto k^0$
(enstrophy equipartition)

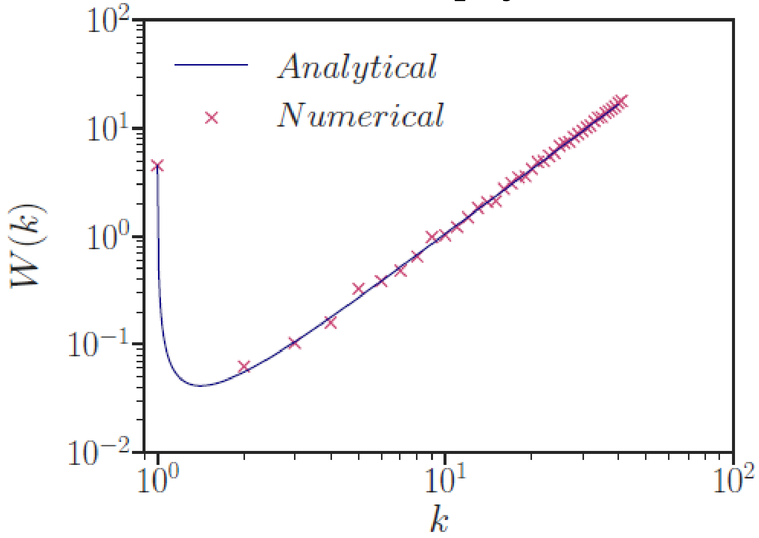
▪ Absolute equilibrium without VS (helical turbulence)

➢ Numerical results:

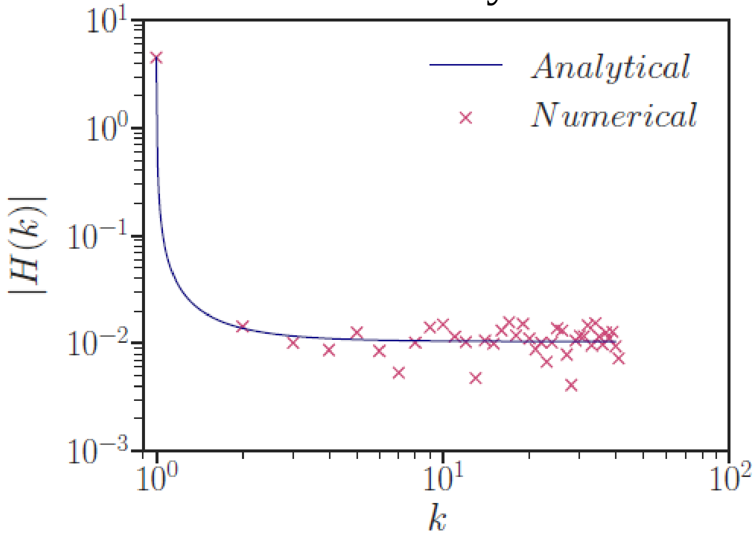
Energy



Enstrophy



Helicity



Once we know the initial conditions (values of enstrophy and helicity), we can predict the final state of an inviscid, truncated, no VS system.

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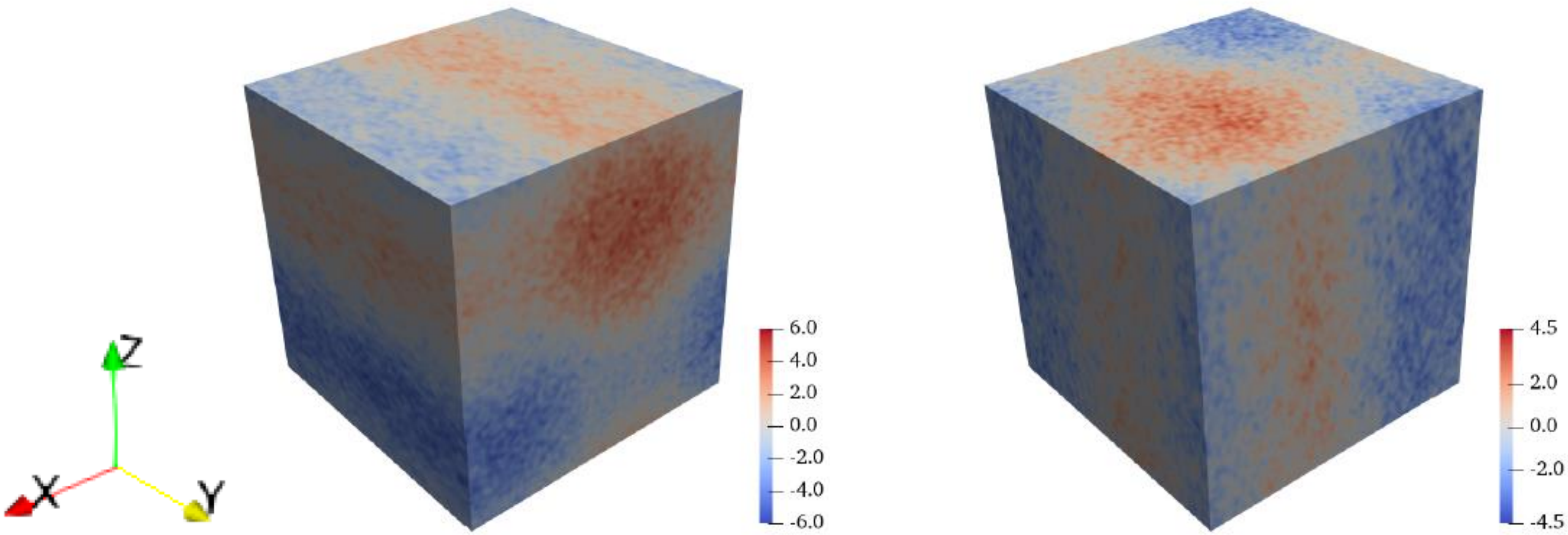
Visualization of the final state

- **Large-scale structure**

  First mode of energy spectrum (87% of the total energy)



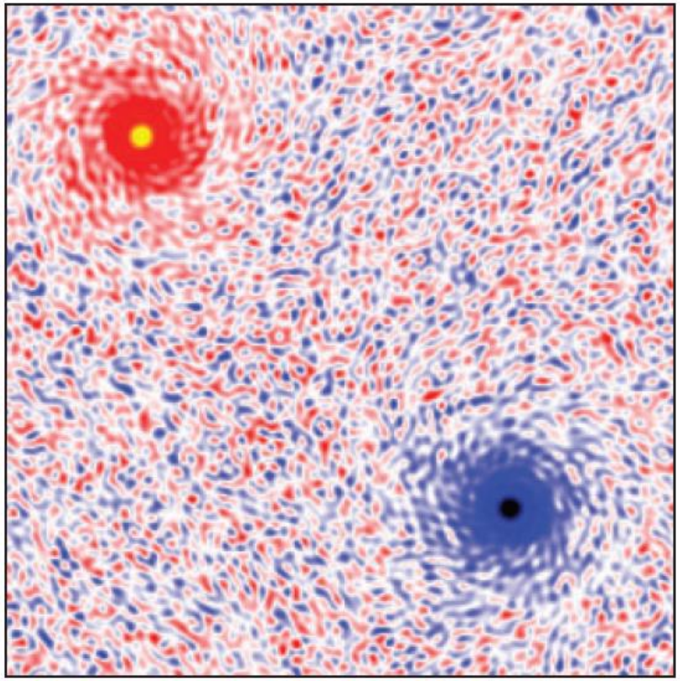
$\boldsymbol{u_y}$ $\boldsymbol{u_z}$

- **ABC flow**

  u_y is almost independent of y, u_z is almost independent of z

- **Compare with 2D turbulence**

Energy Condensation: large-scale structures, acting as a pair of large counter-rotating vortices



Chertkov, M., et al. "Dynamics of energy condensation in two-dimensional turbulence."

Physical review letters 99.8 (2007): 084501.

Final state of 3D no VS turbulence: 3D extension

Viscous turbulence

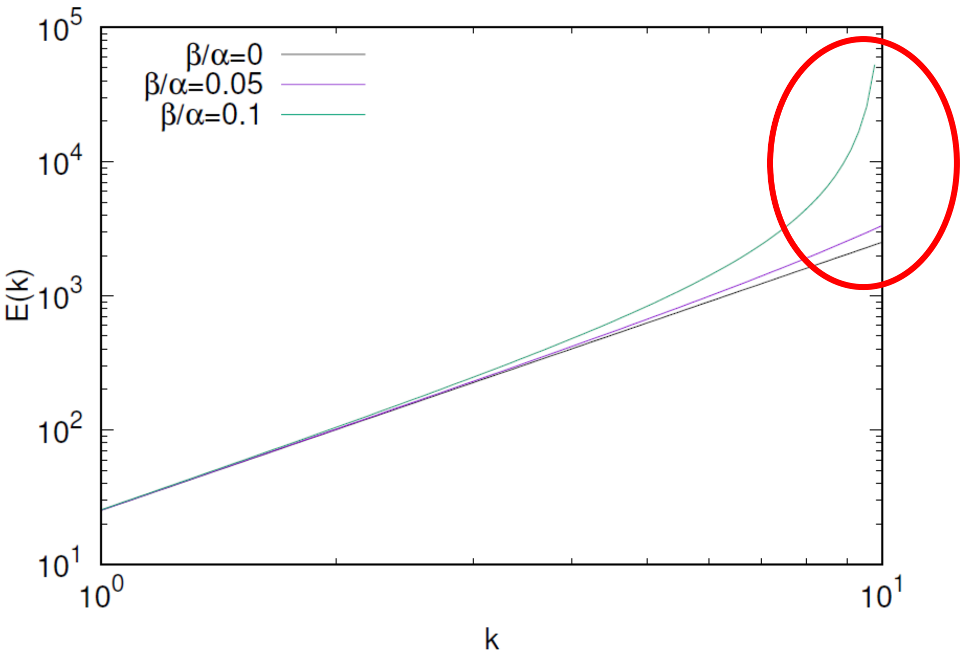
Cascade directions

- **Prediction from the absolute equilibrium state**

➢ Normal 3D turbulence :

Energy transfers from large scales to small scales

Absolute equilibrium state:



$$E(k) = \frac{8\pi\alpha k^2}{\alpha^2 - \beta^2 k^2}$$

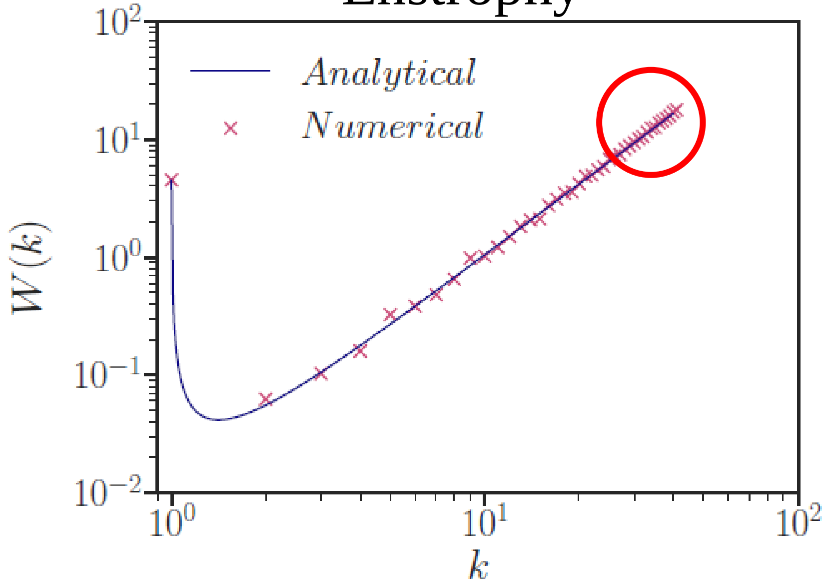
(Kraichnan 1973)

Kraichnan, Robert H. "Helical turbulence and absolute equilibrium." Journal of Fluid Mechanics 59.4 (1973): 745-752.

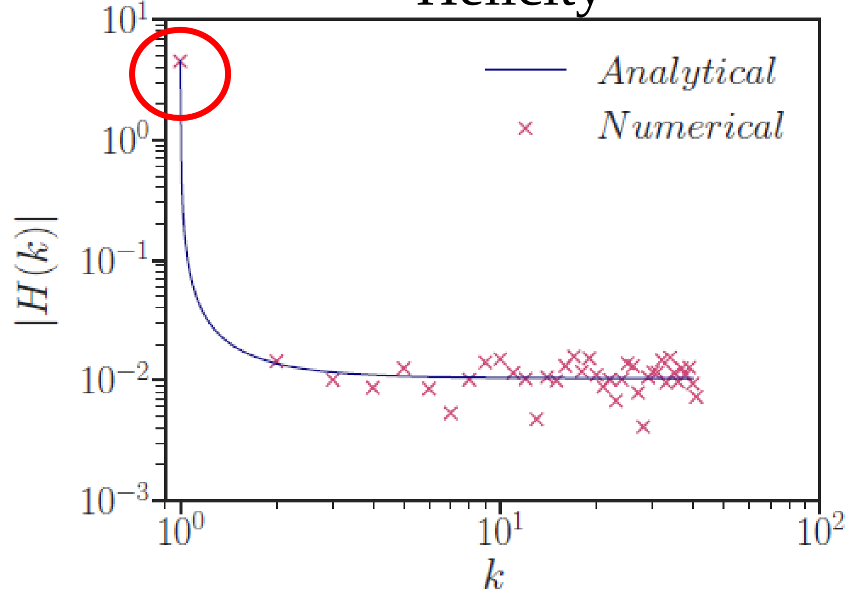
▪ Prediction from the absolute equilibrium state

➢ Turbulence without VS :

Enstrophy



Helicity



- **Dual cascade?**

  Enstrophy: downward cascade   Helicity: upward cascade

- **Energy spectrum?**

  Dimensional analysis: $E(k) \propto \epsilon_{\Omega}^{2/3} k^{\mathbf{-3}}$, $E(k) \propto \epsilon_{H}^{2/3} k^{\mathbf{-7/3}}$

  (ϵ_{Ω}: dissipation rate of enstrophy, ϵ_{Ω}: dissipation rate of helicity)

Numerical simulation

- **Forward enstrophy cascade**

➢ **Numerical setup**

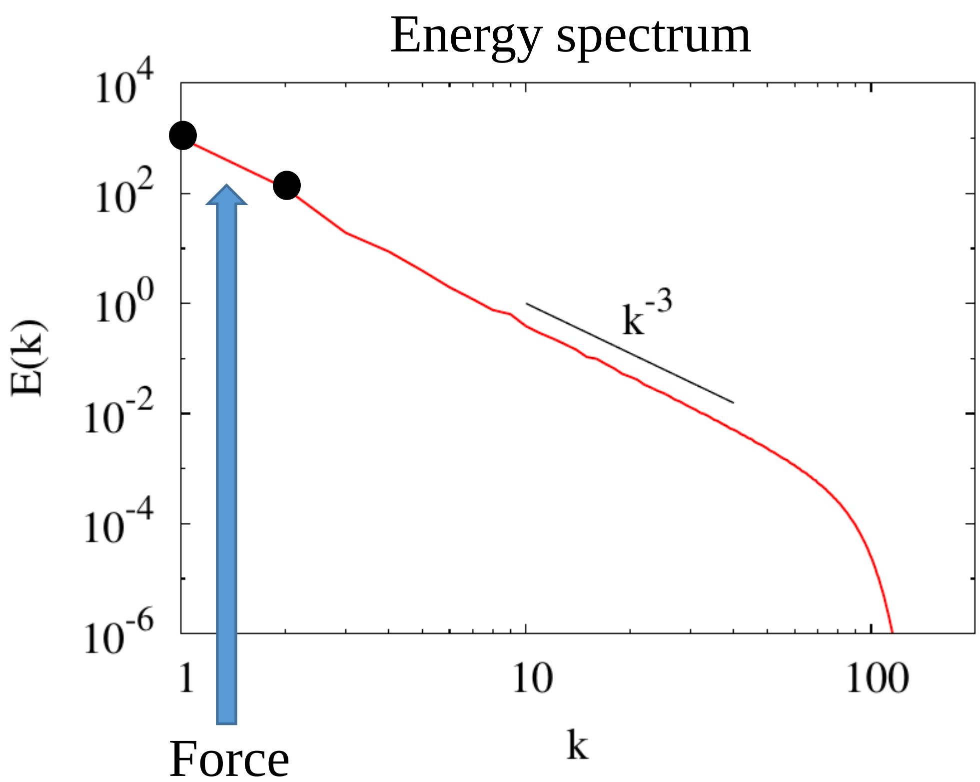
- Forcing (Chen 1993)
- Simulation method

  Spectral method

  DNS

  Grid 512^3
- Inertial range:

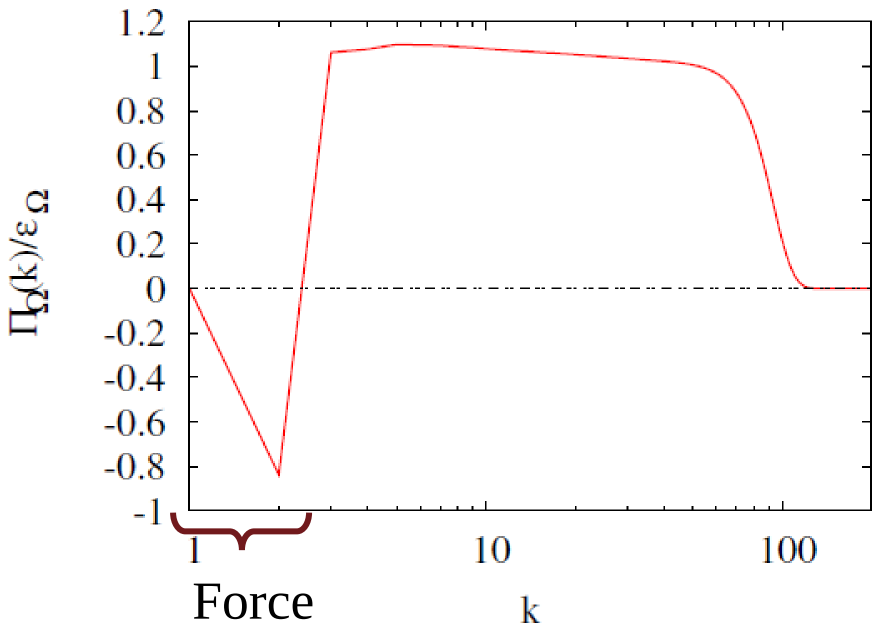
$$E(k) \propto \epsilon_{\Omega}^{2/3} k^{-3}$$

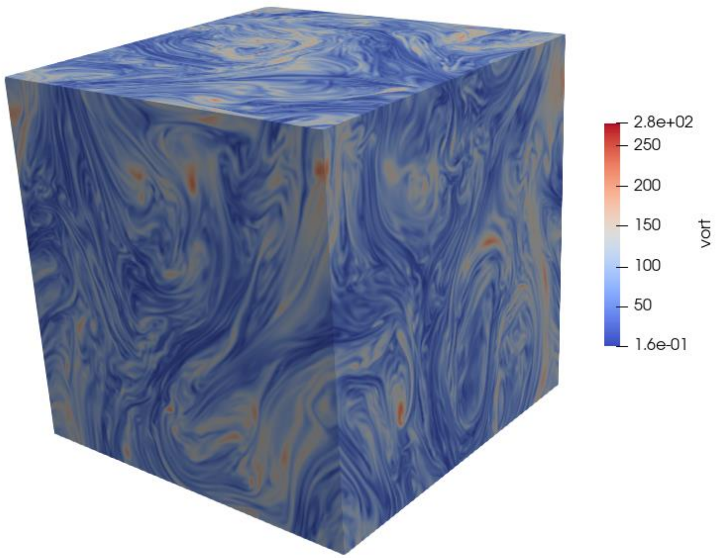


Chen, Shiyi, et al. "On statistical correlations between velocity increments and locally averaged dissipation in homogeneous turbulence." *Physics of Fluids A: Fluid Dynamics* 5.2 (1993): 458-463.

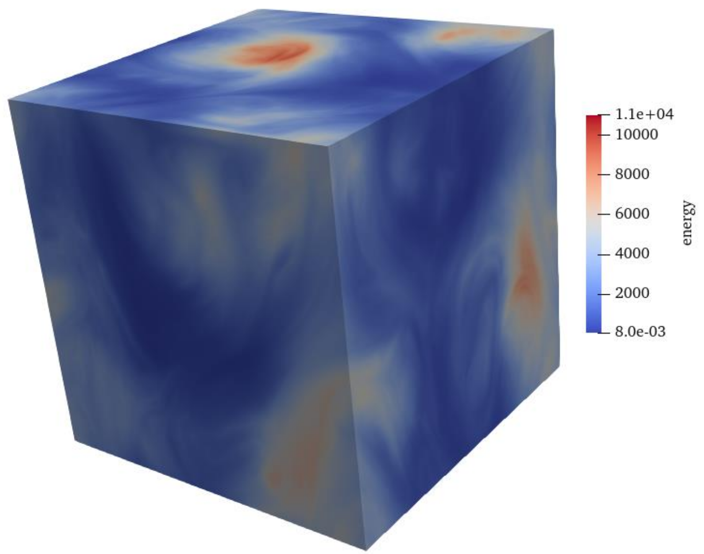
▪ Forward enstrophy cascade

Normalized enstrophy flux





Enstrophy



Energy

- Forward enstrophy cascade
- Conservation of enstrophy

▪ Backward helicity cascade

➢ Numerical setup

- Forcing at $k = 30, 31$
- Grid $256 \times 256 \times 256$
- Hypofriction (Bos 2009)

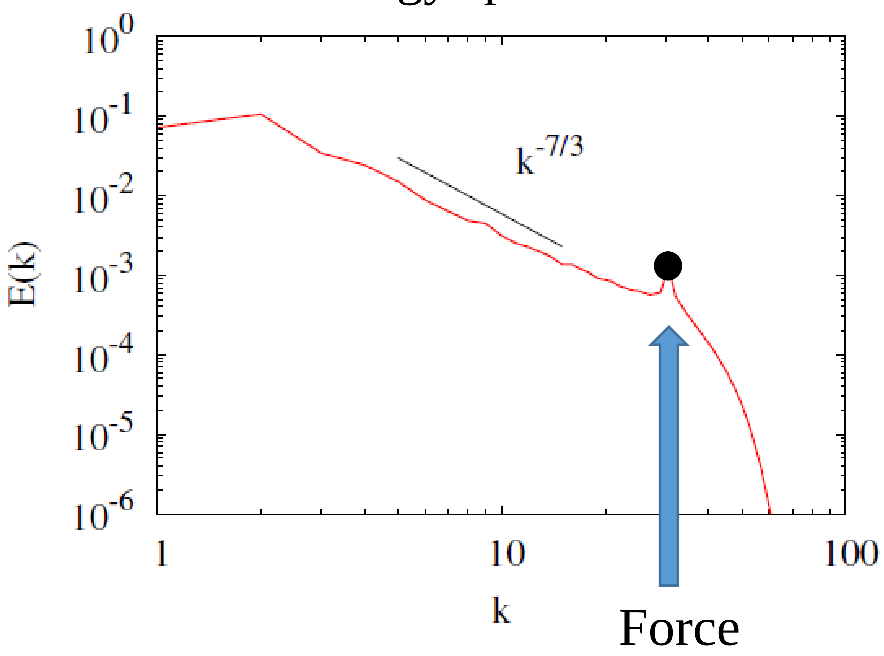
$$\frac{\partial \hat{\boldsymbol{u}}}{\partial t} = -\mu k^2 \hat{\boldsymbol{u}} - \frac{i}{k^2} \boldsymbol{k} \times \mathcal{F}[(\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}]$$

$$+\hat{f} \boxed{- \mu_f \left(\frac{k_f}{k}\right)^{2\alpha} \hat{\boldsymbol{u}}}$$

(with $\alpha = 1$, $k_f = 3$)

Avoid the cumulation of helicity at large scales

Energy spectrum



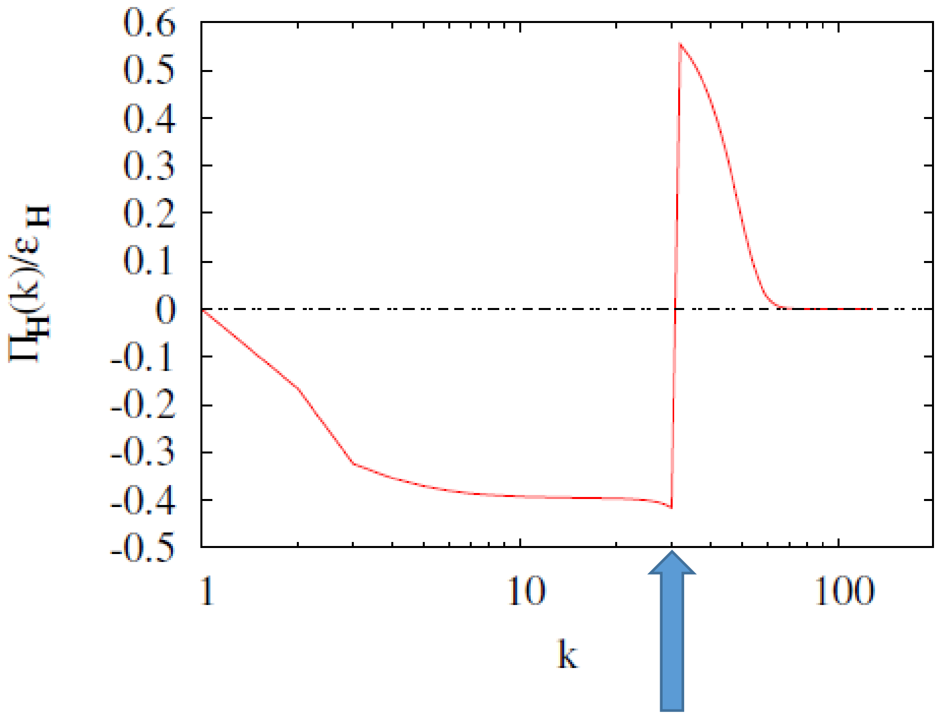
- Inertial range:

$$E(k) \propto \epsilon_H^{2/3} k^{-7/3}$$

Bos, Wouter JT, and Jean-Pierre Bertoglio. "Large-scale bottleneck effect in two-dimensional turbulence." *Journal of Turbulence* 10 (2009): N30.

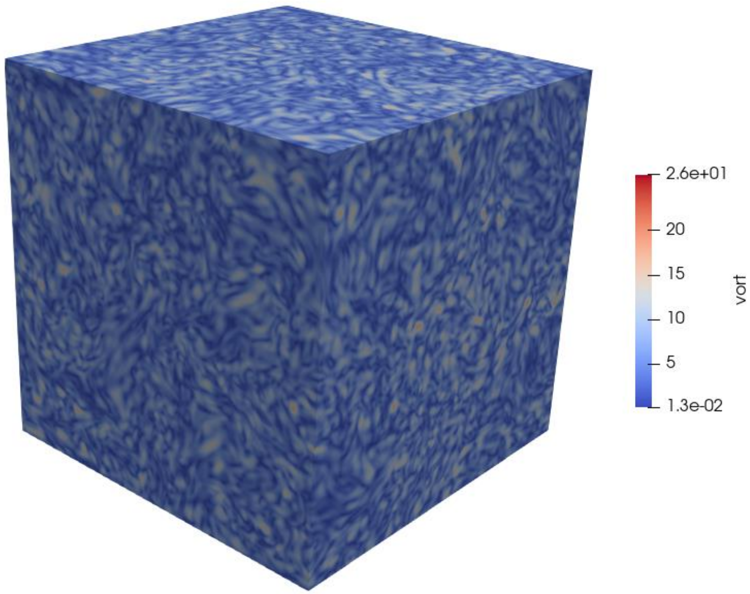
▪ Backward helicity cascade

Normalized helicity flux

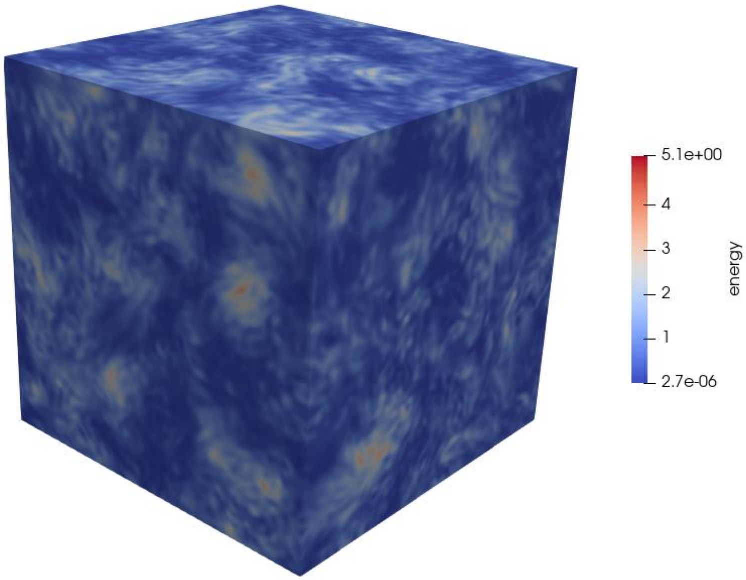


Force

- Backward helicity cascade
- Conservation of helicity



Enstrophy



Energy

Conclusion

- **Inviscid no VS turbulence:**

➢ Absolute equilibrium

Invariant : enstrophy & helicity   Spectra can be predicted

➢ Final state

ABC flow: 3D extension of the final state of 2D turbulence

- **Viscous no VS turbulence:**

➢ Dual cascade

Forward enstrophy cascade & Backward helicity cascade

➢ New slope of energy spectrum

Forward enstrophy cascade: $E(k) \propto k^{-3}$

Backward helicity cascade: $E(k) \propto k^{-7/3}$

Thanks!

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