Finite dissipation from helicity following reconnection

Robert M. Kerr   University of Warwick

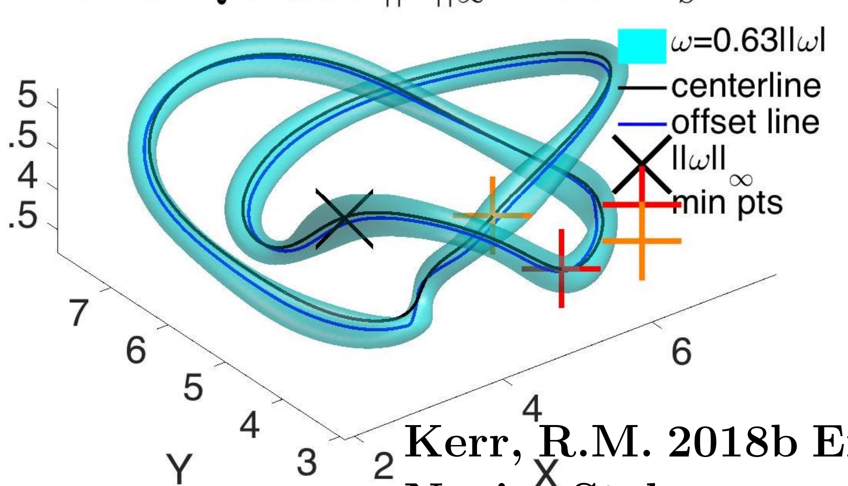
Underlying question:

- If one defines turbulence to be: The fluid motion responsible for the observed finite time, Reynolds number (Re) independent, dissipation of finite energy ΔE.
- And given: Compact initial conditions placed in whole space, infinite $\mathbb{R}^3$.
- Is there a mathematical **barrier** that **blocks** finite time ΔE forming as Re$\to\infty$?
  - Without assuming finite-time singularities for the underlying equations.
- **Further question:** And if this **barrier** exists, as often claimed, **could this be seen numerically** with new diagnostics?

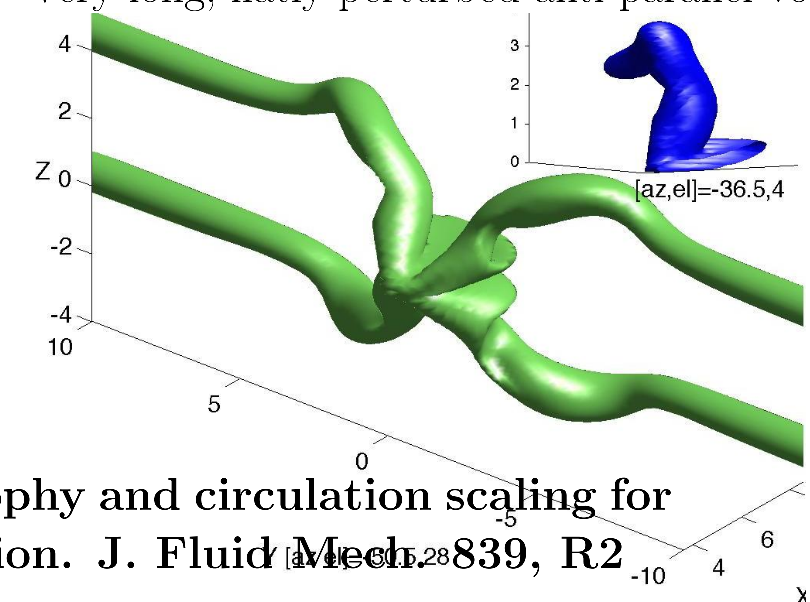
Tool: I will address these questions using two classes of DNS: (Kerr , 2018b)

Strongly perturbed trefoil vortex knot

$t=6$ Q-trefoil $||\omega||_\infty=1.13$ $\mathcal{L}_S=3$



Very long, flatly-perturbed anti-parallel vortices.



Kerr, R.M. 2018b Enstrophy and circulation scaling for Navier-Stokes reconnection. J. Fluid Mech. 839, R2

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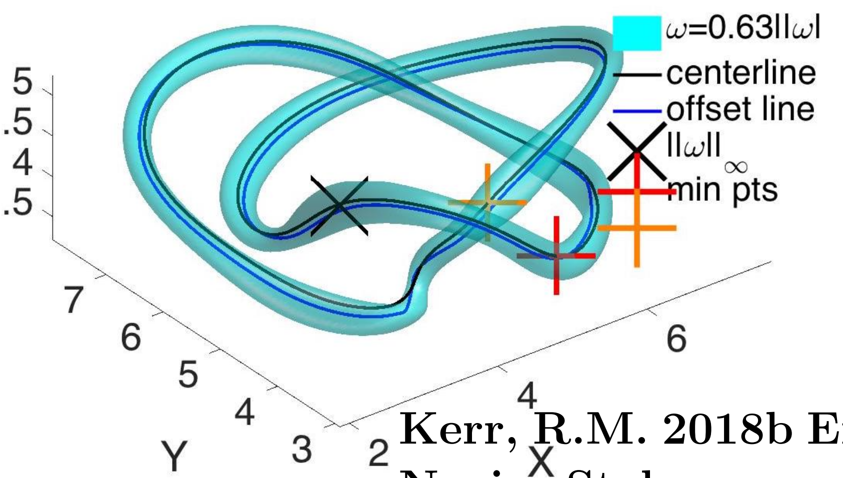

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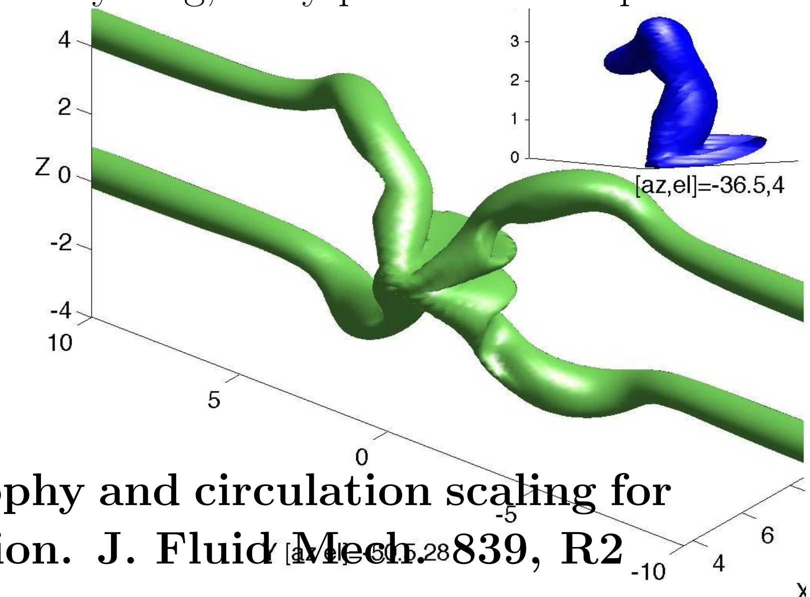

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1. **The new calculations are turbulent by these standards:**

   a) Convergence of energy dissipation ϵ and finite ΔE. b) $k^{-5/3}$ spectra.

2. What has broken the **barrier?**

   a) Answer: **Space.** b) **Why?** c) It's associated with **reconnection ending. How?**

3. Is **helicity** the **barrier**? a) And/Or: Does **helicity** break the **barrier**?

Finite dissipation from helicity following reconnection

Underlying question: Is there a mathematical **barrier** that **blocks**

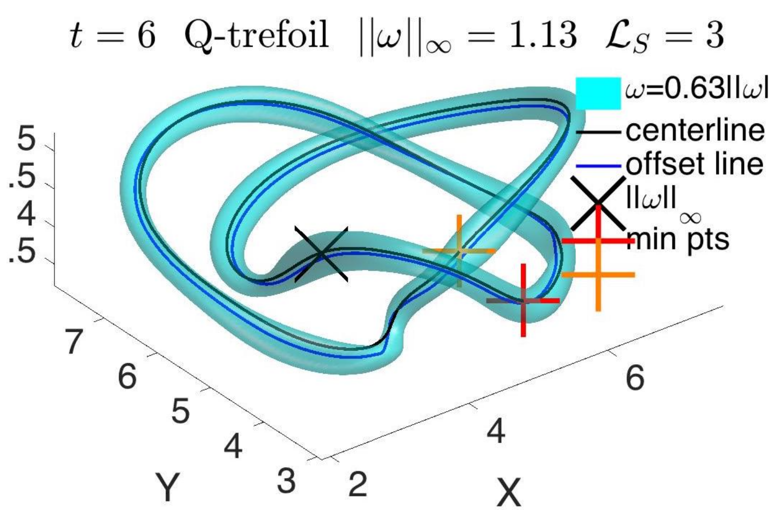
- finite time ΔE $Re \to \infty$ forming? To address with **new** extensions of Kerr (2018b).

1. **The new calculations are turbulent by these standards:**
   a) Convergence of energy dissipation ϵ and finite ΔE.
   b) And $k^{-5/3}$ spectra.
2. What has broken the **barrier?**
   a) Answer: **Space.** b) **Why?** c) It's associated with **reconnection ending. How?**
3. Is **helicity** the **barrier**?
   a) And/Or: Does **helicity** break the **barrier**?

Existing work/presentations:
Kerr, R.M. 2018b Enstrophy and circulation scaling for Navier-Stokes reconnection. J. Fluid Mech. 839, R2
See: https://video-archive.fields.utoronto.ca/view/10320

Trefoil Experiment

Initial condition



$\Omega = \ell^3$ is domain size.

How did I get started on non-anti-parallel?

$\Leftarrow$ Inspiration:: **Helicity conservation by flow across scales in reconnecting vortex links and knots** Proc. Nat. Acac. Sci. **111** (2014). Scheeler, D. K., D. P., G. L. K., W. T. M. Irvine.

Primary diagnostics: **Enstrophy** and **Helicity**

$$\boldsymbol{Z(t)} = \int_{\mathbb{T}^3_\ell} \boldsymbol{\omega}^2 \, dV \quad \mathbf{H(t)} = \int_{\mathbb{T}^3_\ell} \boldsymbol{u}\cdot\boldsymbol{\omega} \, dV,$$

Meaning: Enstrophy $\Rightarrow$ dissipation $\boldsymbol{\epsilon = \nu Z}$

Meaning: Helicity: Knottedness. Global topological number?

Trefoils have three advantages.

1) Their helicity.

2) Reconnection

3) FOR MATHS: Compact initial state:

- Ideal for investigating relationships between reconnection, helicity and finite-time dissipation $\Delta E = \int_0^T \epsilon \, dt$, **T** finite.

Primary diagnostics: **Enstrophy** and **Helicity**

$$Z(t) = \int_{\mathbb{T}_\ell^3} \boldsymbol{\omega}^2 \, dV \quad \mathcal{H}(t) = \int_{\mathbb{T}_\ell^3} \boldsymbol{u}\cdot\boldsymbol{\omega} \, dV,$$

Meaning: Enstrophy $\Rightarrow$ dissipation $\boldsymbol{\epsilon = \nu Z}$

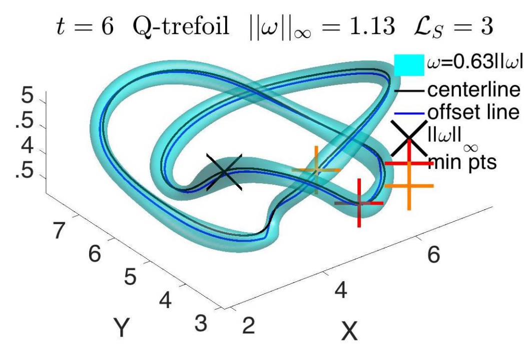
Meaning: Helicity: Knottedness.
Global topological number?

Trefoils have three advantages.

1) **Almost maximal helicity.**

2) **Reconnections rapid.**

3) FOR MATHS: **Compact initial state:**
Can be isolated in $\mathbb{R}^3$.



Caveat

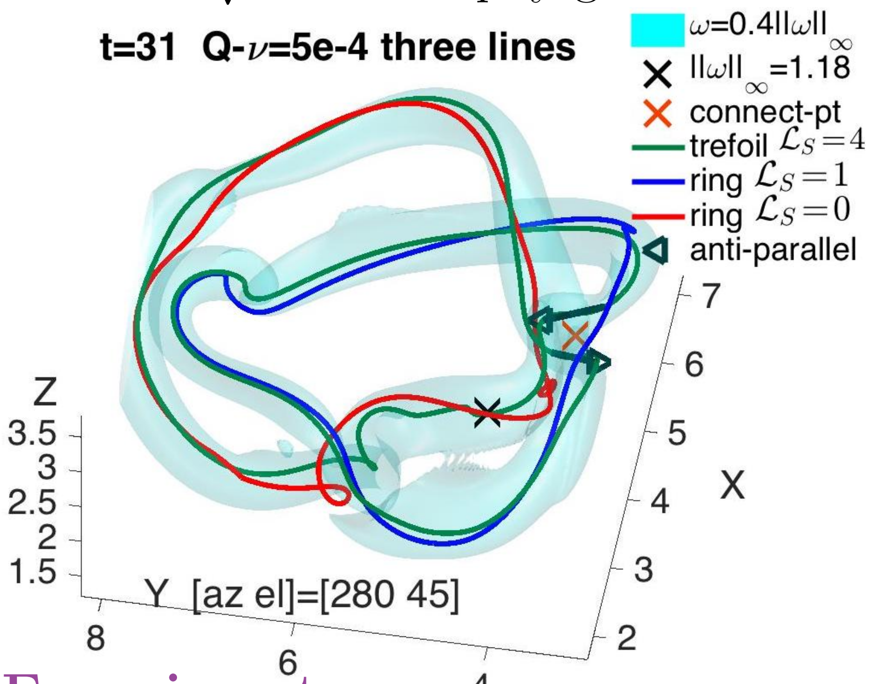
- **Ideal** for investigating relationships between reconnection, helicity and finite-time dissipation $\boldsymbol{\Delta} E = \int_0^T \epsilon \, dt$, **T** finite.

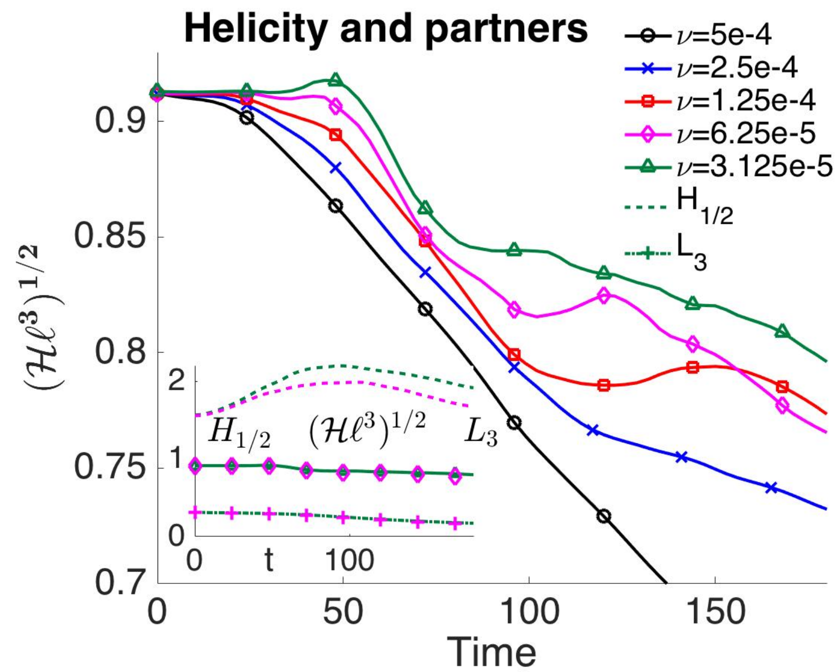
Caveat One absolutely needs **strongly perturbed trefoils**.

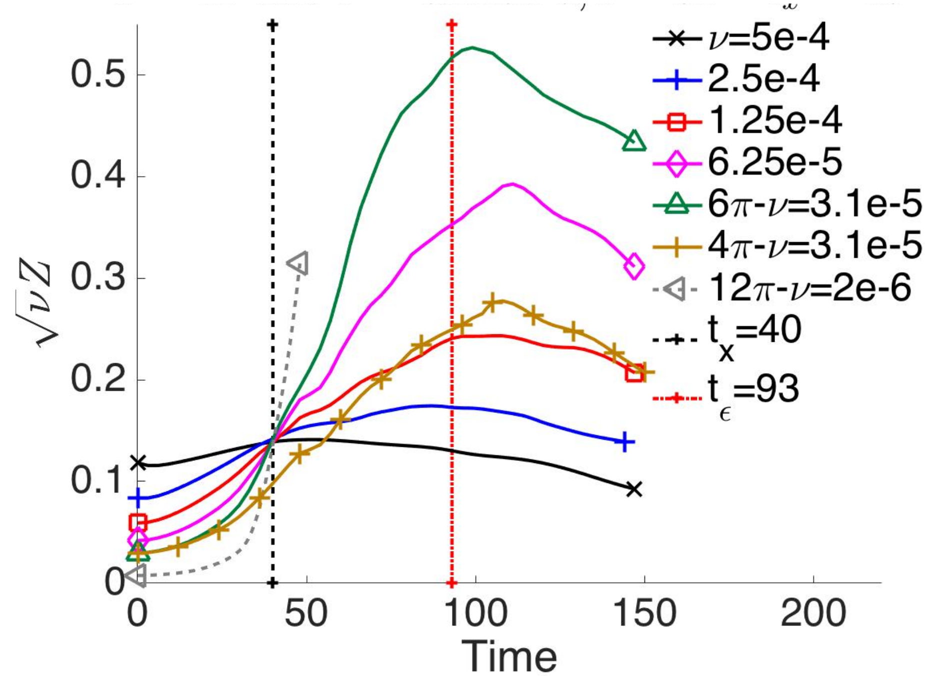
- **Reason** the advantages **fail if true, three-fold symmetric trefoils** are used.

Status from Kerr (2018b):

- Consider $\mathcal{H}^{1/2}$ (helicity)$^{1/2}$ decay and modified $\sqrt{\nu}Z$ enstrophy growth.







Experiments:
Helicity preservation was claimed.

- The first dynamics are associated with $t_x \approx 41$.

- Helicity grows at t_x.(Why?)

- Convergence of $\sqrt{\nu}Z$ **at** t_x. (Why?)

$t_x \approx 41$ can now be associated with reconnection ending.

Euler, **Navier-Stokes** on cubic torus $\mathbb{T}^3_\ell$,
i.e. ℓ^3 periodic domains. The vorticity $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$

$$\frac{\partial \boldsymbol{u}}{\partial \mathbf{t}} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla\mathbf{p} + \nu\triangle\boldsymbol{u} \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial \mathbf{t}} + (\boldsymbol{\omega}\times\boldsymbol{u}) = -\nabla\mathbf{p_h} + \nu\triangle\boldsymbol{u} \tag{2}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} + \nu\triangle\boldsymbol{\omega}, \qquad \nabla\cdot\boldsymbol{\omega} = 0\,. \tag{3}$$

the enstrophy density $|\boldsymbol{\omega}|^2$ equation and the volume-integrated enstrophy Z are

$$\frac{\partial|\boldsymbol{\omega}|^2}{\partial t} + (\boldsymbol{u}\cdot\nabla)|\boldsymbol{\omega}|^2 = \underbrace{2\boldsymbol{\omega}\boldsymbol{S}\boldsymbol{\omega}}_{Z_p=\text{production}} + \nu\triangle|\boldsymbol{\omega}|^2 - \underbrace{2\nu(\nabla\boldsymbol{\omega})^2}_{\epsilon_\omega=Z-\text{dissipation}}\,, \qquad Z = \int \boldsymbol{\omega}^2 dV\,, \tag{4}$$

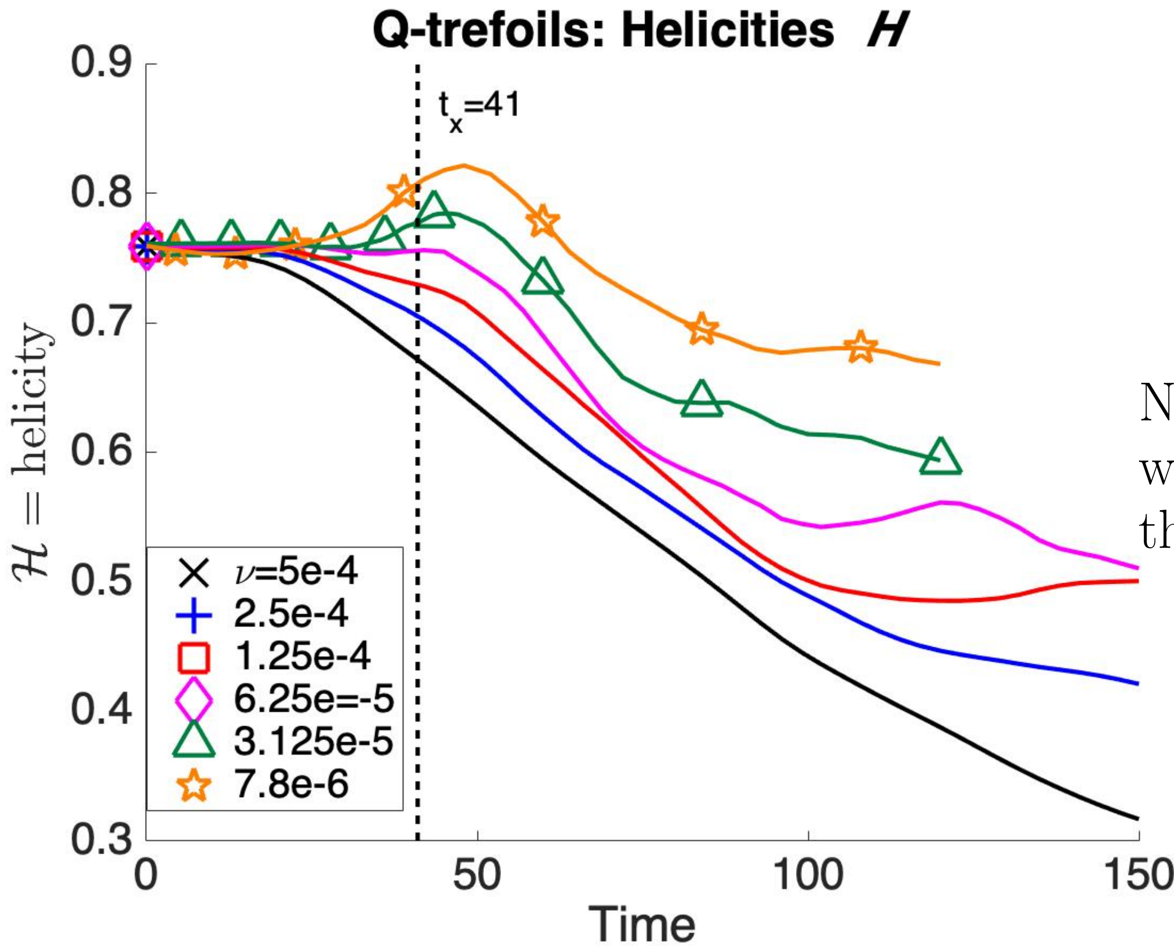
and the helicity density h equation and global helicity $\mathcal{H}$ are

$$\frac{\partial h}{\partial t} + (\boldsymbol{u}\cdot\nabla)h = \underbrace{-\boldsymbol{\omega}\cdot\nabla\Pi}_{\omega-\text{transport}} + \underbrace{\nu\triangle h}_{\nu-\text{transport}} - \underbrace{2\nu\text{tr}(\nabla\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}^T)}_{\epsilon_h=\mathcal{H}-\text{dissipation}} \qquad \mathcal{H} = \int \boldsymbol{u}\cdot\boldsymbol{\omega}dV\,. \tag{5}$$

where $\Pi = p - \frac{1}{2}\boldsymbol{u}^2 \neq p_h$ is not the pressure head $p_h = p + \frac{1}{2}\boldsymbol{u}^2$.

Further diagnostics related to helicity. Scaled to have units of circulation.

- Sobolev norm of order 1/2: $H_\ell^{(1/2)}$ • Square-root of helicity $\mathcal{H}^{1/2}$.
- Cubic velocity normed: L_3. • L_3 independent of ν implies Navier-Stokes regularity.



New $\nu = 7.8$e-6 calculation with a **greater increase** in the helicity $\boldsymbol{\mathcal{H}}$.

$$\frac{\partial h}{\partial t} + (\boldsymbol{u}\cdot\nabla)h =(5) \underbrace{-\boldsymbol{\omega}\cdot\nabla\Pi}_{\omega-\text{transport}} + \underbrace{\nu\triangle h}_{\nu-\text{transport}} - \underbrace{2\nu\text{tr}(\nabla\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}^T)}_{\epsilon_h=\mathcal{H}-\text{dissipation}} \qquad (5)$$

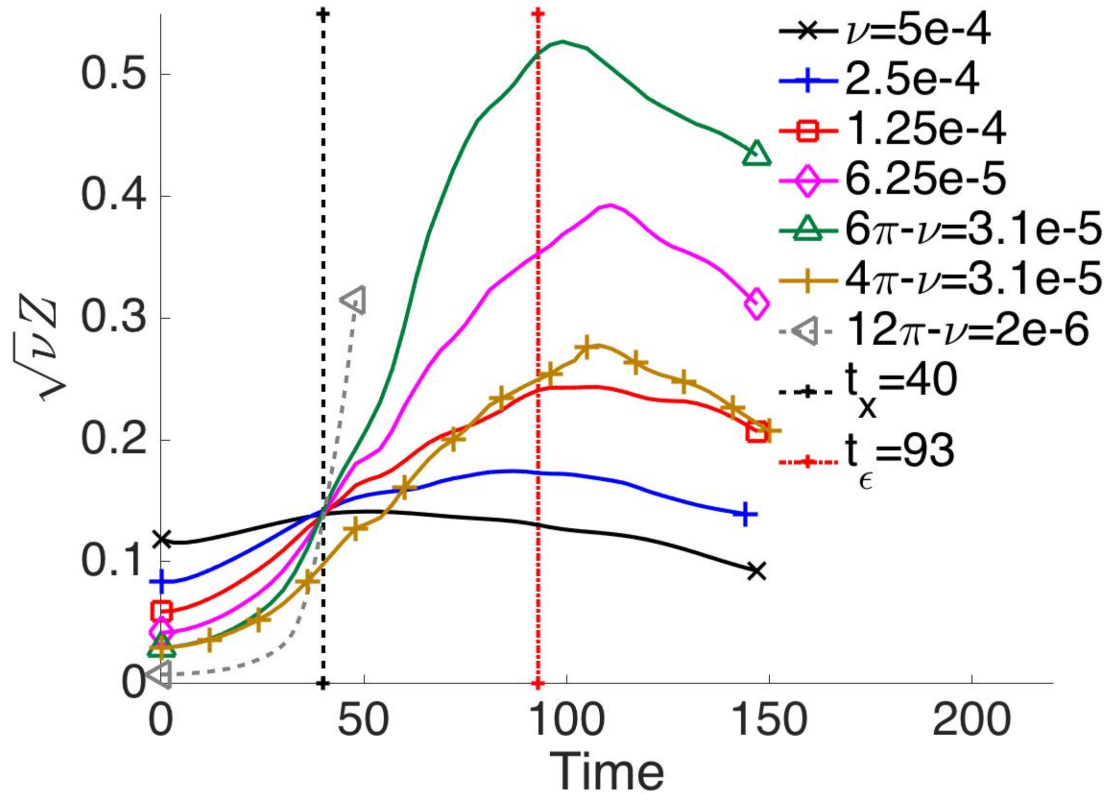
For: $\Pi = p - \frac{1}{2}\boldsymbol{u}^2 \neq p_h$ (not $p_h = p + \frac{1}{2}\boldsymbol{u}^2$.)

- Viscous-ν-transport is usually insignificant.
- Vorticity-$\boldsymbol{\omega}$-transport **could be significant**.
- Only **dissipation** of $+h$ or $-h$ can change $\mathcal{H}$.

My new diagnostics:
Colour-contour vorticity isosurfaces with $-\boldsymbol{\omega}\cdot\nabla\Pi$.
(and $\mathcal{H}$-transfer-spectra)

- **Caveat** on convergence of $\sqrt{\nu}Z$ scaling at t_x.

$\ell = 4\pi$ but $\nu = 3.125\text{e-}5, \ell = 6\pi \quad t_x = 40$



- **True only so long as** the domain **size ℓ grows as ν decreases.**
  - Purpose of the extra **brown +** curve:
  - **Brown +** calculation uses the same small viscosity $\nu = 3.125e-5$ as the **green curve**,
  - but the domain is smaller, $(\mathbf{4}\pi)^3$ not $(\mathbf{6}\pi)^3$. **Why?**

(Also Gibbon EPL (2020))

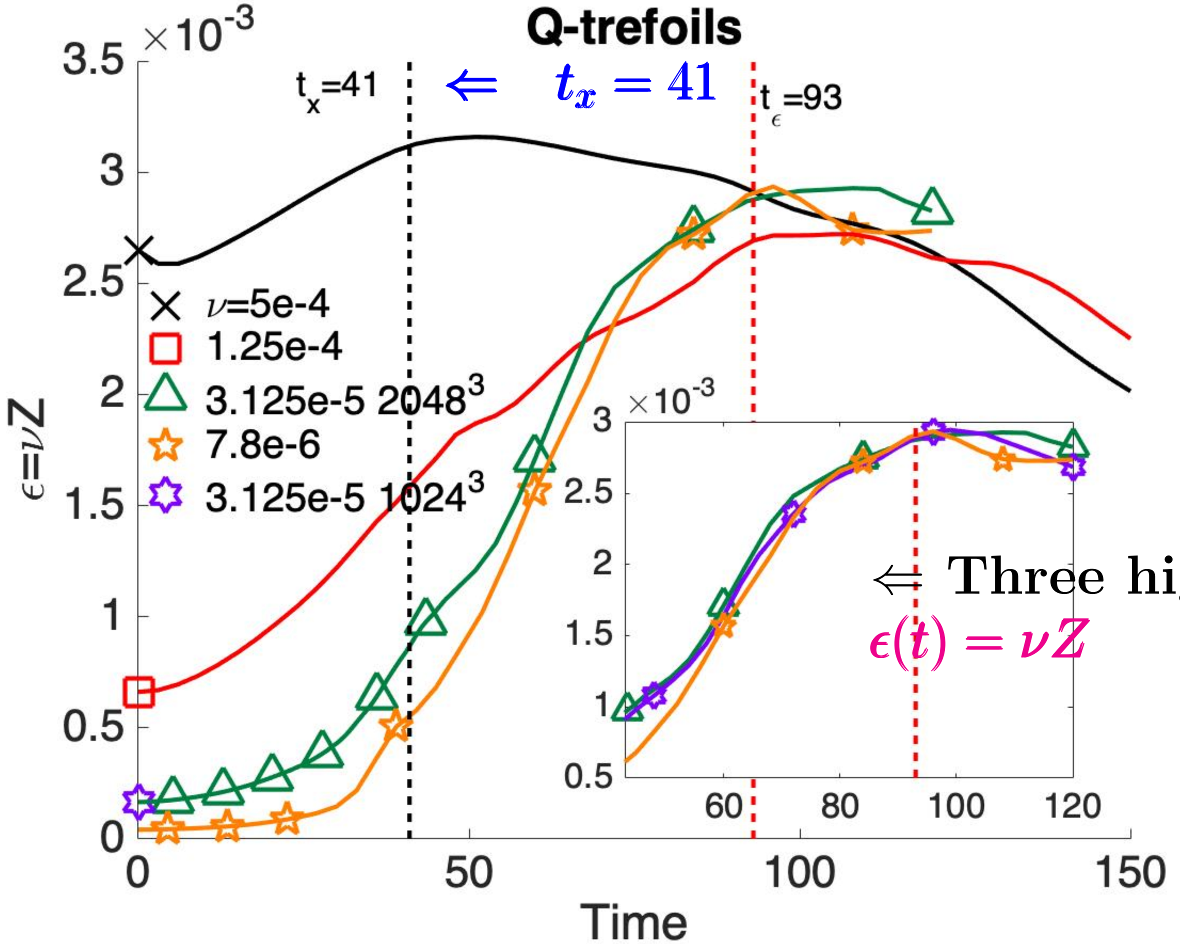
* A partial answer comes from **Constantin CMP (1986) "Note on Loss of Regularity for Solutions of the 3D Incompressible Euler and Related Equations".**

- **With post-1986 C&F88 Maths.**
- **Both Z & $\epsilon = \nu Z$**
- **are bounded in**
- **fixed domains**
- **if Euler is regular.**

But what if the domain is not fixed?

Next: Dissipation $\epsilon = \nu Z$

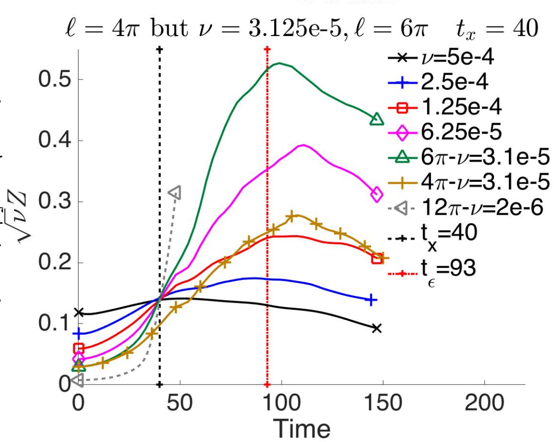
- What happens to the dissipation $\epsilon = \nu Z$ after the $\sqrt{\nu} Z$ cross at $t = t_x \approx 41$?



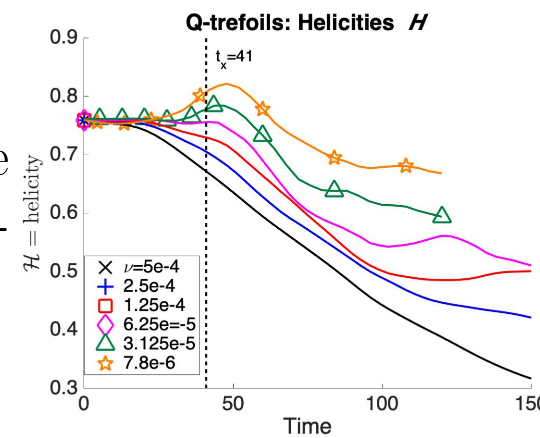
$\Leftarrow$ **Three highest Reynolds numbers.**

$\epsilon(t) = \nu Z$

- Increase in enstrophy is spawned by the $t \approx 41$ end of the first reconnection. $\Rightarrow$



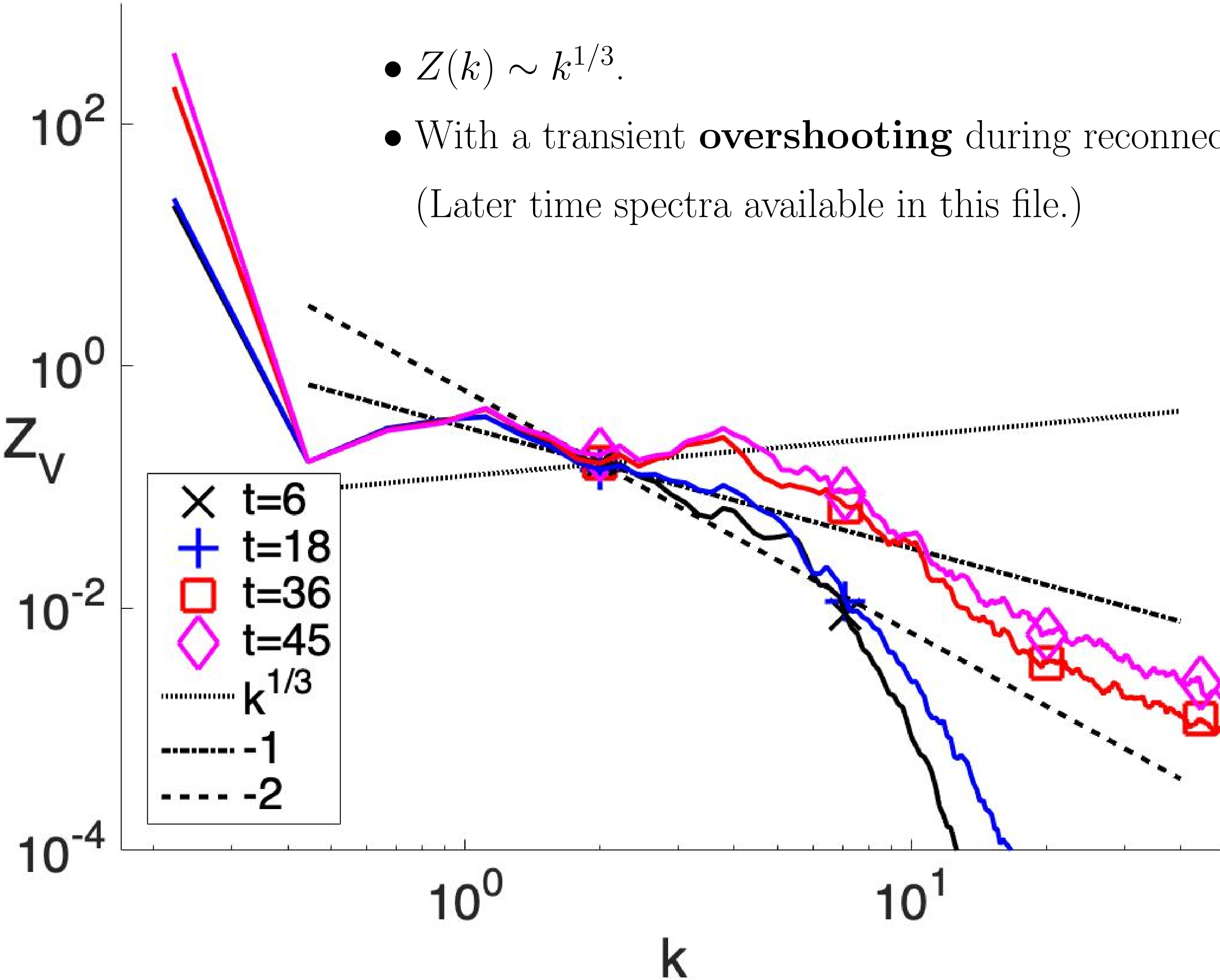
- Helicity increase also due to reconnection? $\Rightarrow$



Enstrophy 3D-spectra case Q-ν=7.8e-6

- $Z(k) \sim k^{1/3}$.
- With a transient **overshooting** during reconnection.

(Later time spectra available in this file.)

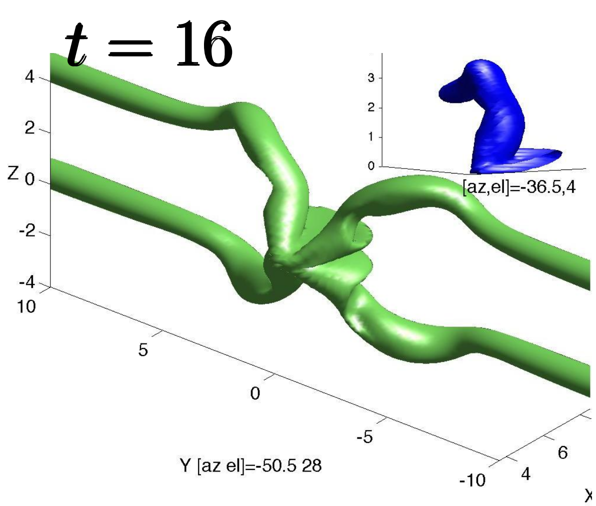


Is there any $k^{1/3}$ (1/3=-5/3+2) regime?

Very long, flatly-perturbed anti-parallel pair.

$t < 14$ **Preliminary: Before reconnection.**

Reconnection: Secondary vortices.



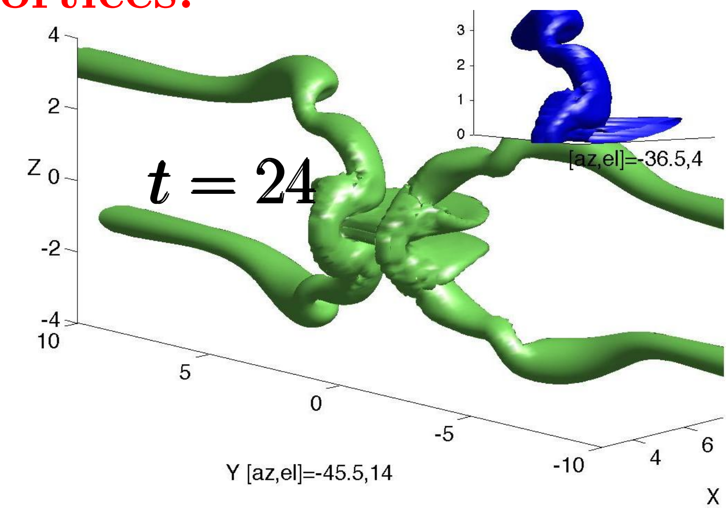
⇐ Pair touch, form **tent**. (next)

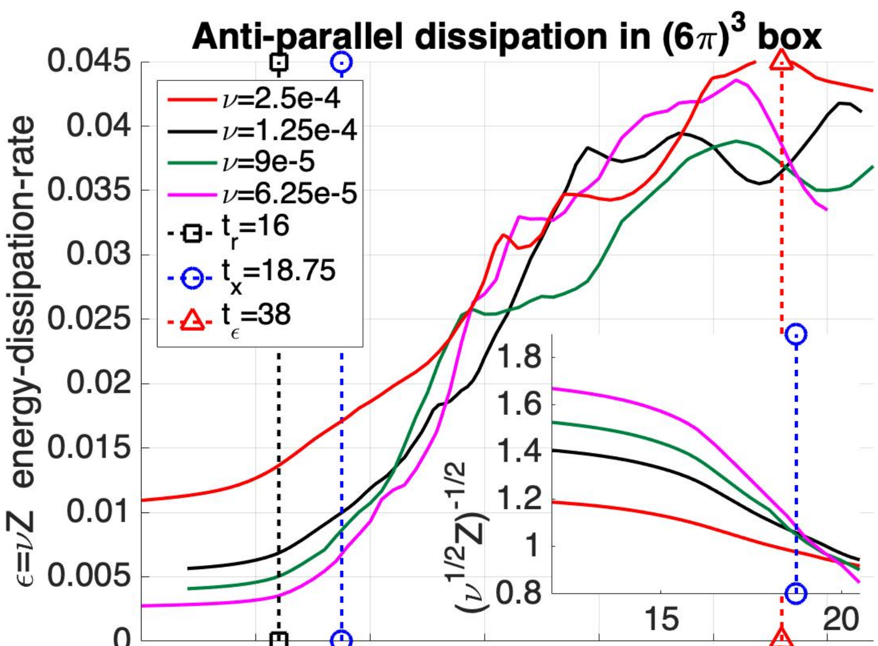
Reconnected pair, separate and twist. ⇒

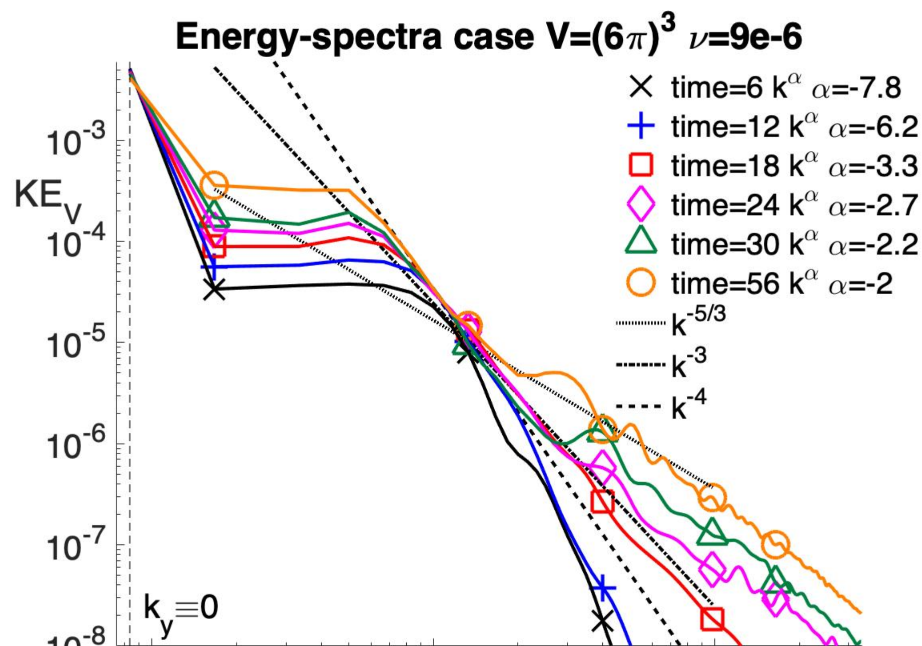
Low Reynolds=5000

Details in Kerr (2018b).

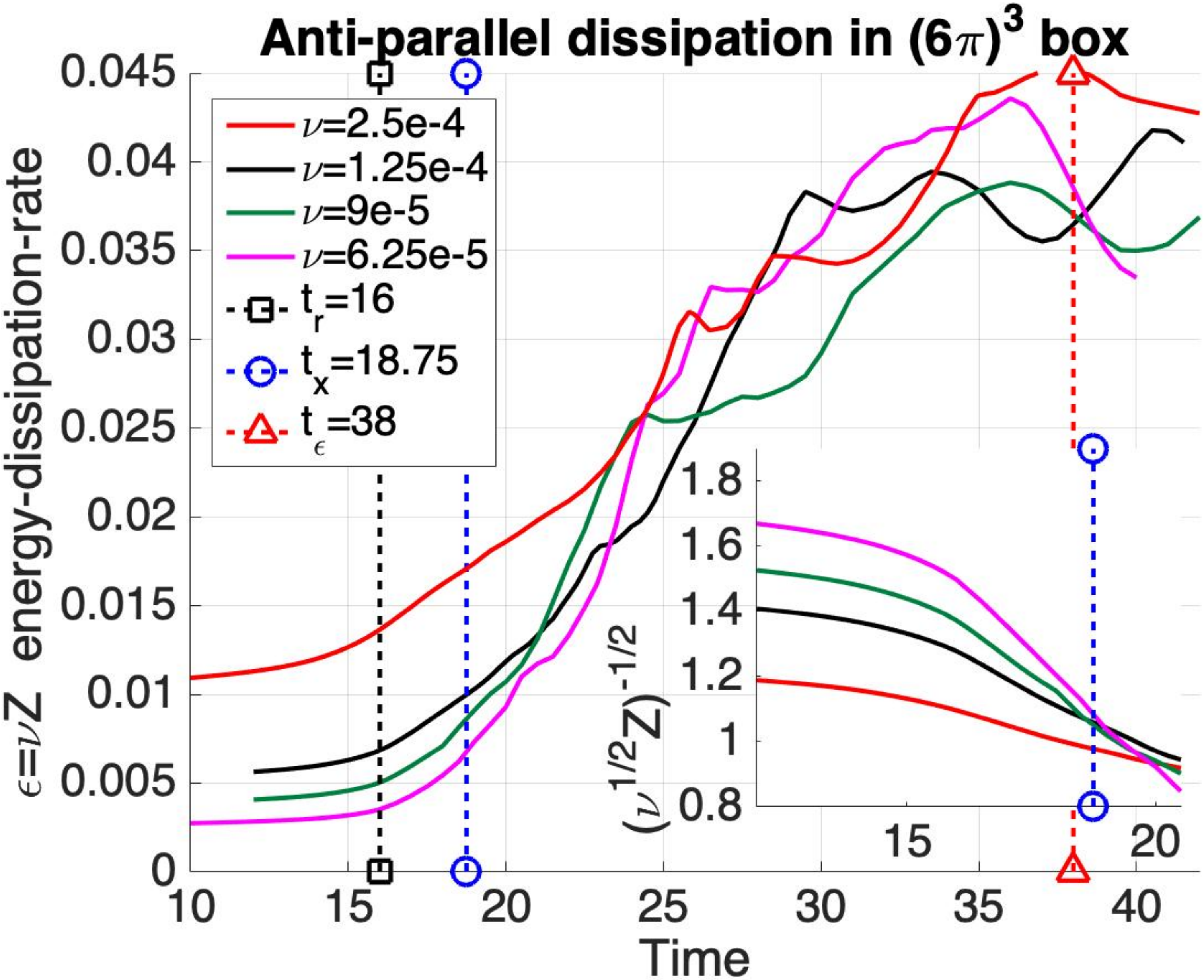
New: $Re = \Gamma/\nu = 80,000$.







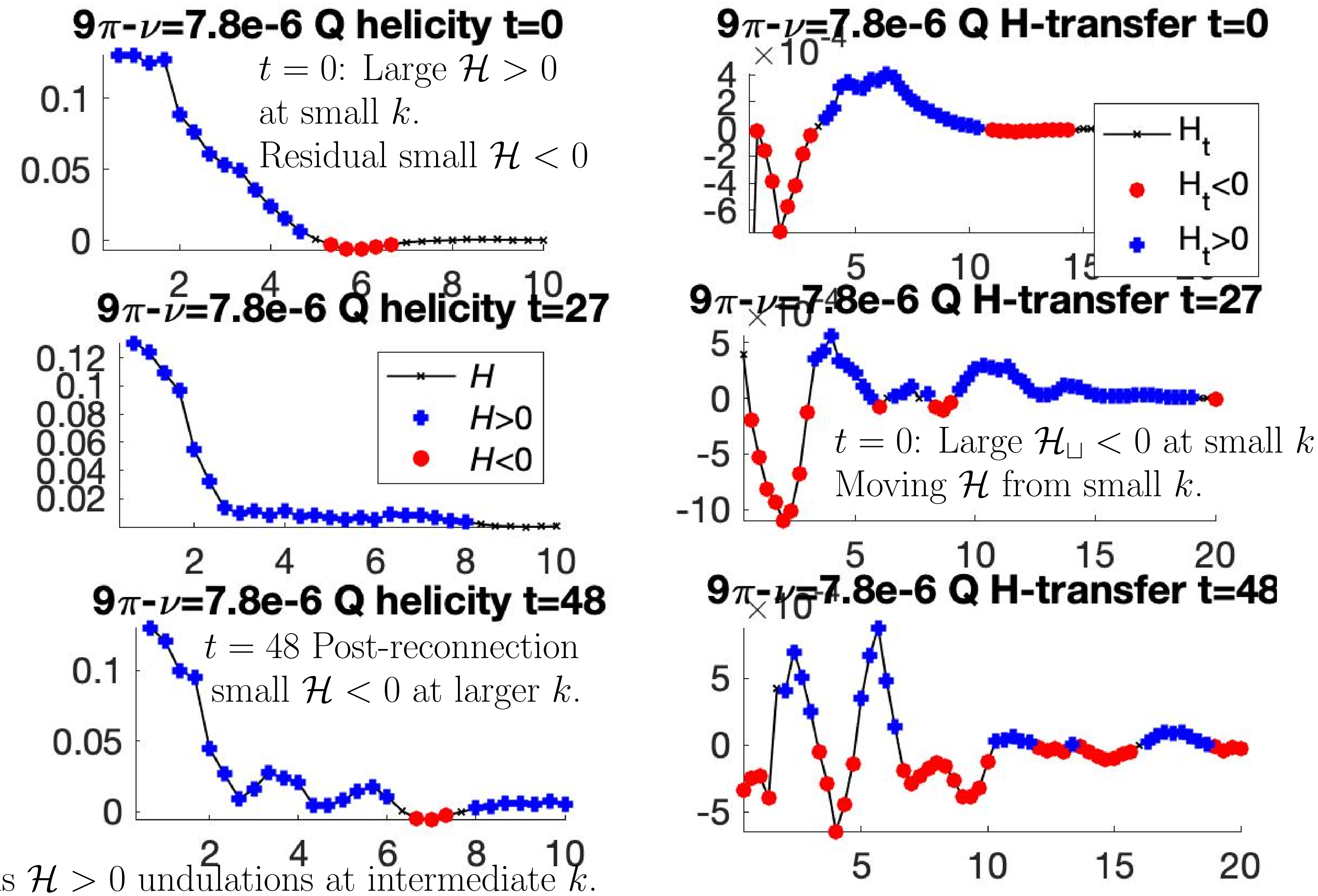
Would like to **contrast** these results to: McKeown et al (2019); Yao & Hussain (2020). But will just discuss the $\epsilon(t)$ convergence for now.



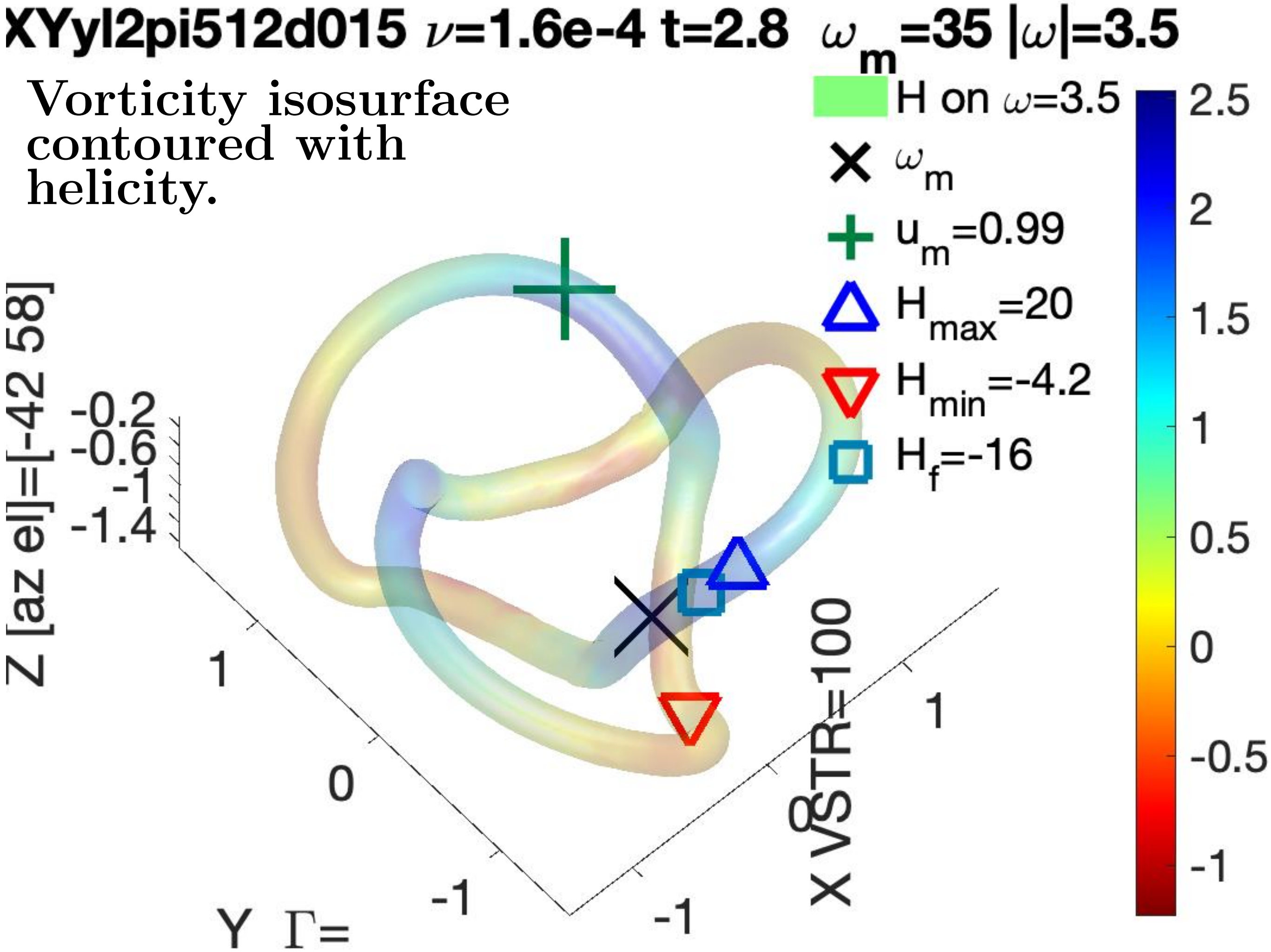
Would like to **contrast** these $(6\pi)^3$ results to: McKeown et al (2019); Yao & Hussain (2020); But will just discuss the $\epsilon(t)$ convergence for now.

See: https://video-archive.fields.utoronto.ca/view/10320 (April 2019 Fields Institute)

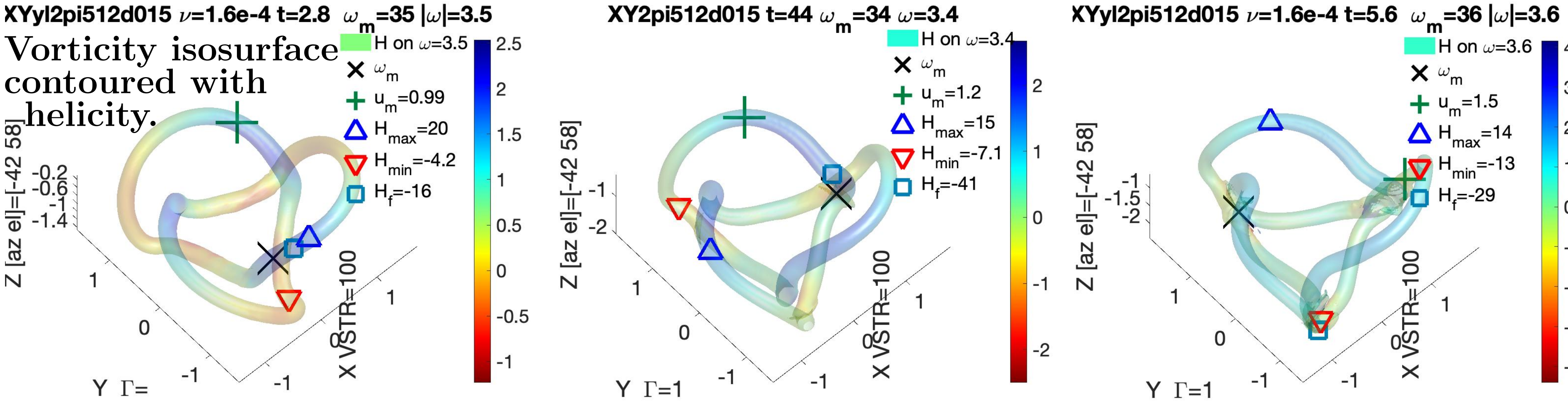
BACK TO TREFOILS



- These spectra suggest some type **shell model dynamics?**
- Since this is **3D-DNS**, could **physical space fluxes help** in our understanding?



$t = 2.8$ is an **unperturbed** trefoil with the **three-fold symmetry** and **thinner core**.
Different timescales and **suppresses** the evolution with enstrophy initially decaying. So the helicity density $\boldsymbol{h}$ **is still mostly the original positive value** along the vortex.



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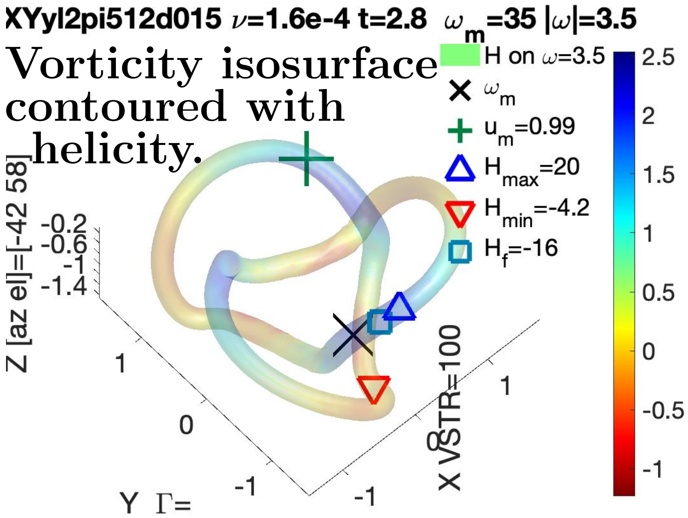
$t = 4.4$

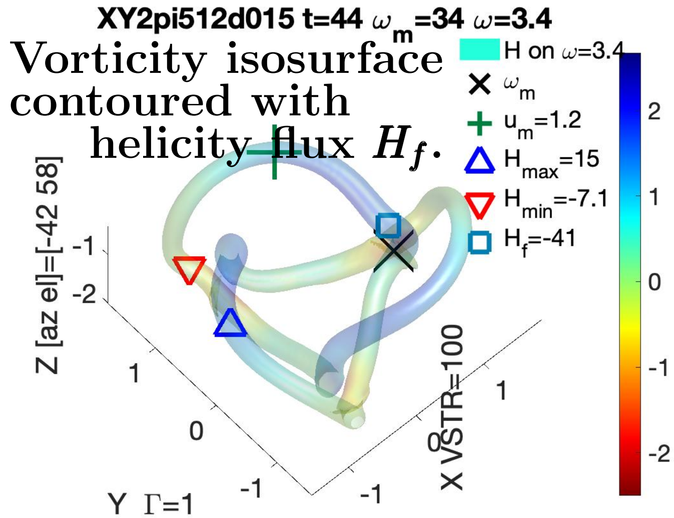
Distortions before reconnection.

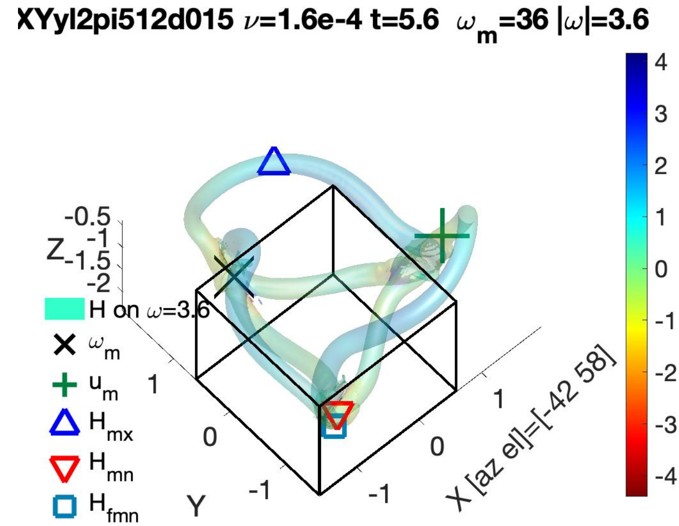
Vorticity isosurface contoured with helicity flux $\boldsymbol{H_f = -\omega \nabla \Pi}$**.**

$t = 5.6$

During reconnection.







$t = 2.8$ is an unperturbed trefoil with the **three-fold symmetry**. This suppresses the evolution with enstrophy initially decaying. So the helicity density ***h* is still mostly the original positive value** along the vortex.

$t = 4.4$

- **x** ω_m
- $+$ u_m
- △ $\max(\mathcal{H})$
- ▽ $\min(\mathcal{H})$
- □ $\min(\boldsymbol{\omega}\cdot\nabla\Pi)$
- x $\max(\epsilon_Z)$
- ◁ $\min(Z_p)$
- ▷ $\max(Z_p)$

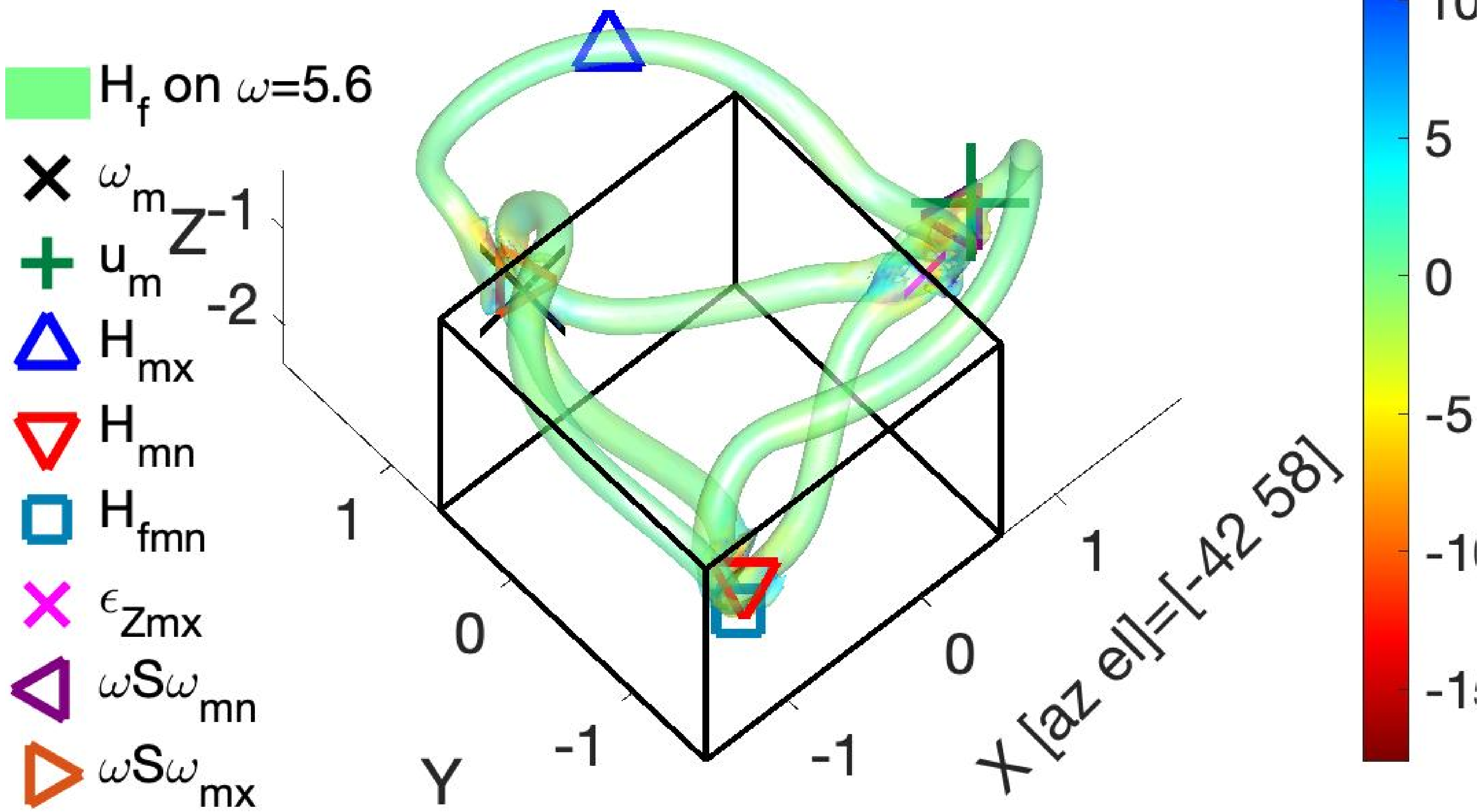
$t = 5.6$

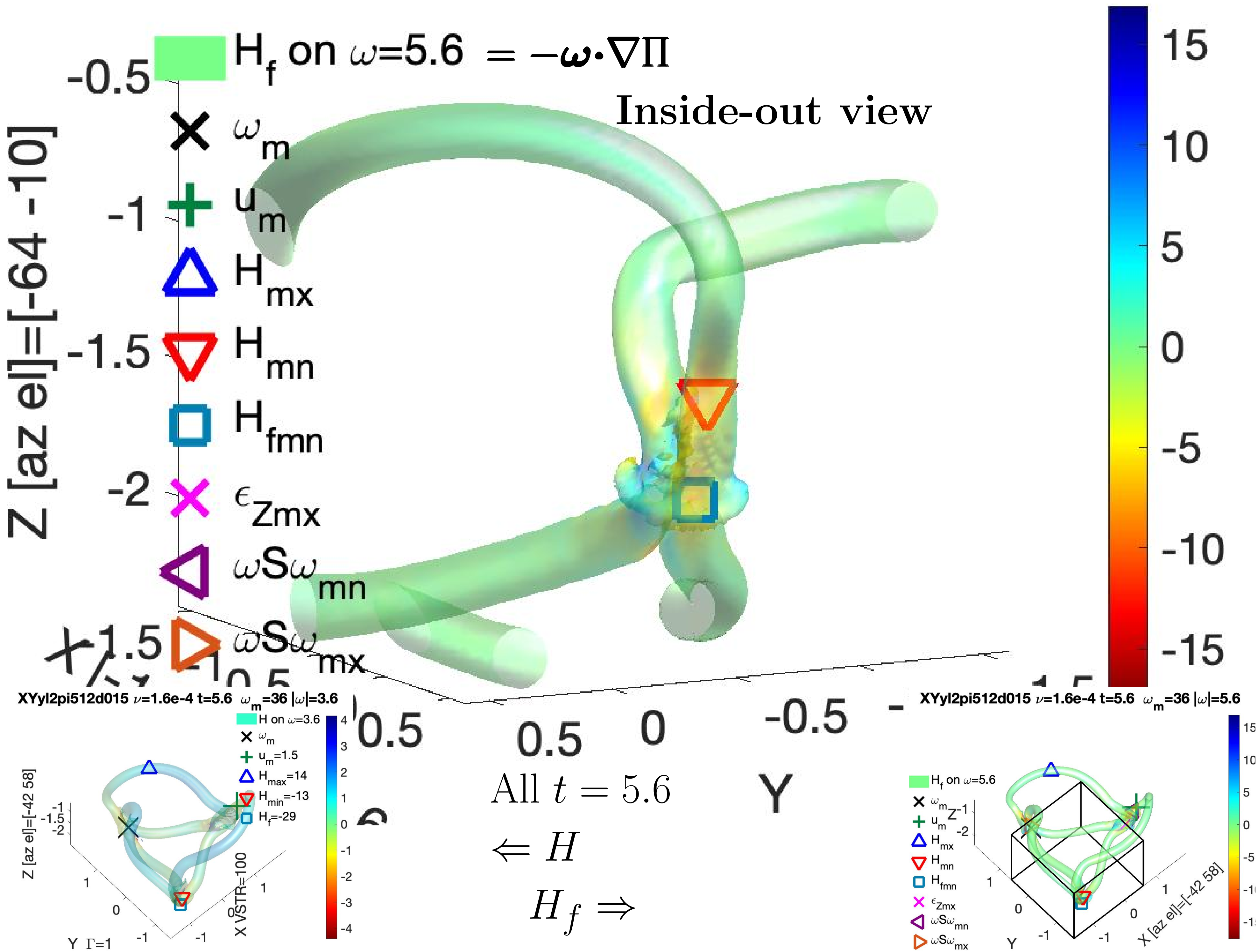
During reconnection

- All **maxes** and **mins** are along the vortex and, except for the velocity maximum (green $+$), are **clustered** around the **tipping points**. Those points where reconnection will eventually occur.
- **Now we focus on tipping point in the x=y=-1 corner.**

XYyl2pi512d015 ν=1.6e-4 t=5.6 ω_m=36 $|\omega|$=5.6

Note that all maxes and mins (except max $\mathcal{H}$) are clustered around the tipping points.
The focus now will be on the tipping point highlighted.
The view will be from the inside out. Helicity flux $\boldsymbol{H_f} = -\boldsymbol{\omega}\nabla\Pi$.





H_f on ω=5.6 = −ω·∇Π
Inside-out view
Z [az el]=[-64 -10]
ω_m
u_m
H_mx
H_mn
H_fmn
ε_Zmx
ωSω_mn
ωSω_mx
Y
All t = 5.6
⇐ H
H_f ⇒
XYyl2pi512d015 ν=1.6e-4 t=5.6 ω_m=36 |ω|=3.6
H on ω=3.6
u_m=1.5
H_max=14
H_min=-13
H_f=-29
Z [az el]=[-42 58]
Y Γ=1
X VSTR=100
XYyl2pi512d015 ν=1.6e-4 t=5.6 ω_m=36 |ω|=5.6
H_f on ω=5.6
X [az el]=[-42 58]

What has been shown:

1. Extensions in time and Reynolds numbers of the two calculations of Kerr (2018b).

- A strongly perturbed trefoil vortex knot. • Long, perturbed anti-parallel vortices.

   a) **New:** Both **now** have convergent dissipation rates $\boldsymbol{\epsilon(t)}$ as $\nu \to 0$.

   b) $k^{-5/3}$ **inertial range** spectra.

   c) *Dissipation anomalies*, $\Delta E = \int_0^T \epsilon\, dt$=**finite-dissipation in a fixed finite-time** T, are generated **without singularities** or roughness as $\boldsymbol{\nu \to 0}$.

2. **Unresolved mysteries from Kerr (2018b).**

   a) **Why** $\sqrt{\boldsymbol{\nu}}\boldsymbol{Z(t)}$ **converges** at the **reconnection time** $\boldsymbol{t_x}$.

   b) **Why** does $\boldsymbol{\mathcal{H}}$ **increase at** $\boldsymbol{t_x}$.

   c) **Provide: Physical understanding** for **why** increasing the domain size as viscosity $\boldsymbol{\nu}$ **decreases** is necessary for maintaining $\sqrt{\boldsymbol{\nu}}\boldsymbol{Z(t)}$ **convergence** .
   (To complement the Mathematical reason based upon Constantin (1986).)

3. **Answers??**

   a) **A new role for negative** $\mathcal{H} < 0$ **transported to the inner regions??**

   b) **Dissipation of this** $\mathcal{H} < 0$ **leads to an increase in helicity.**

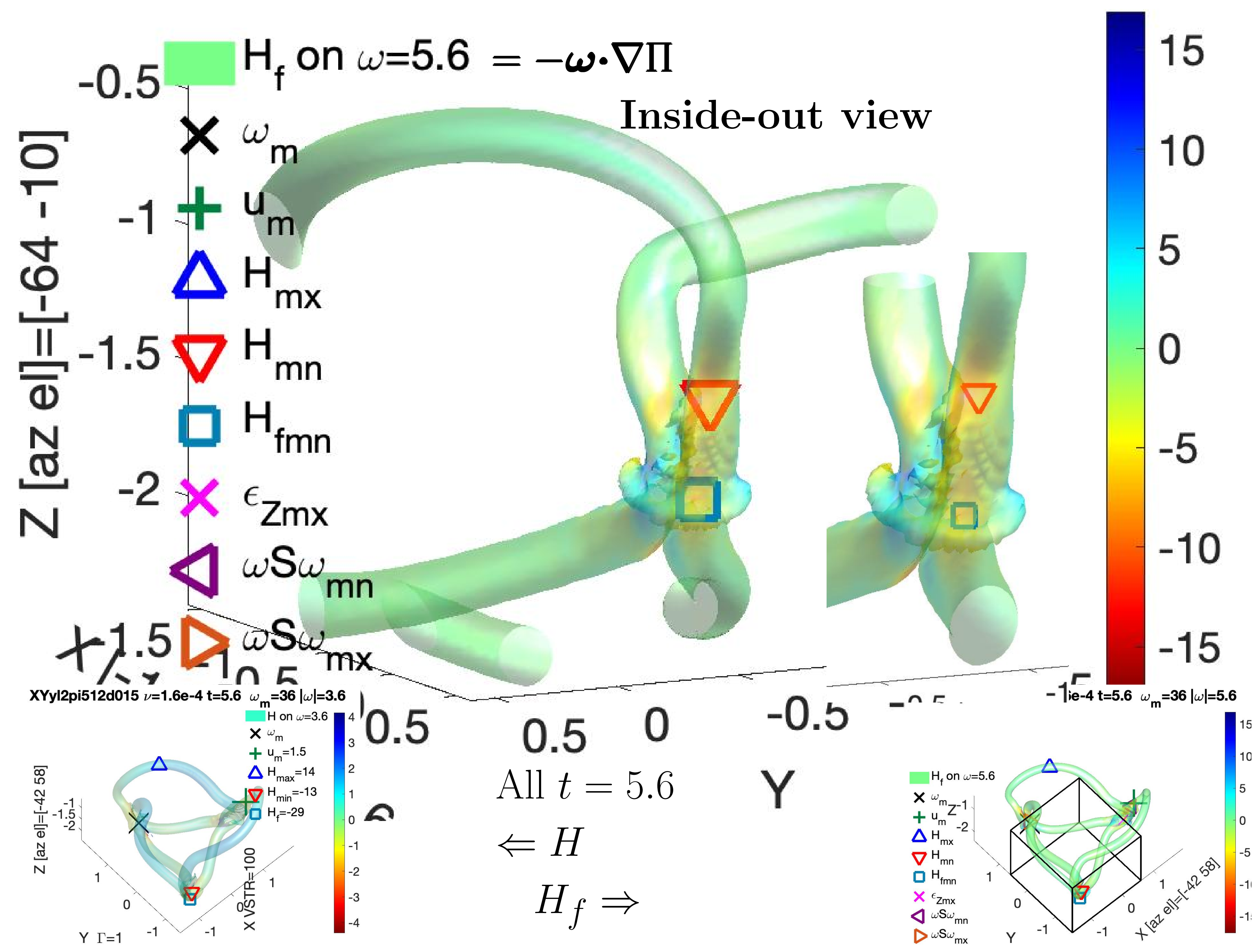
   c) **Which would** remove the barrier from $\boldsymbol{\mathcal{H} > 0}$ which was **blocking** the **growth of small-scale enstrophy** Z **and dissipation** $\boldsymbol{\epsilon}$.

Goals accomplished?

My new calculations are extensions in time and Reynolds numbers of the calculations in Kerr (2018b), a strongly perturbed trefoil vortex knot and very long, flatly-perturbed anti-parallel vortices and the goal is to extend those calculations into a regime with both convergent $\epsilon(t)$ as ν decreases and a $k^{-5/3}$ inertial range spectrum at $t_\epsilon \approx 2t_x$.

A common feature of both sets of calculations is that they are done in very large periodic domains, with the domain size increasing as the viscosity ν decreases. The reasons given Kerr (2018b) are based upon the mathematics of Constantin (1986), mathematics that can also help explain why symmetric initial states can never provide the answers. To accommodate the large domains, the vortices of the anti-parallel set are very long with a localised perturbation. For the trefoil knots, the initial helical vortex state is confined within small volume, so it is mathematically compact.

Besides tracking the convergence of $\epsilon(t)$ as $\nu \rightarrow 0$ and spectra, the exchange of circulation convergence in the early reconnection phase is improved and there is new helicity analysis using the helicity flux along vortex tubes and spectra. The figures below are from an extension of the $\nu = 7.8 \times 10^{-6}$, $Re = \Gamma/\nu = 64,000$ trefoil with $t_x \approx 41$ Kerr (2018b). During the trefoil's reconnection phase the global helicity barely changes, but both the physical and Fourier space analysis shows negative helicity being transported to small scales, which is compensated for by positive helicity moving to large scales. Then, during viscous reconnection, most of small-scale negative helicity is dissipated. Leaving behind an global increase in the positive helicity. The strong perturbation is essential for achieving these results, as three-fold symmetric trefoils suppress $\epsilon(t)$ growth much as close, periodic boundaries can suppress such growth for anti-parallel vortices.



H_f on ω=5.6 = −ω·∇Π
Inside-out view
Z [az el]=[-64 -10]
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H_mn
H_fmn
ε_Zmx
ωSω_mn
ωSω_mx
Y
All t = 5.6
⇐ H
H_f ⇒
XYyl2pi512d015 ν=1.6e-4 t=5.6 ω_m=36 |ω|=3.6
H on ω=3.6
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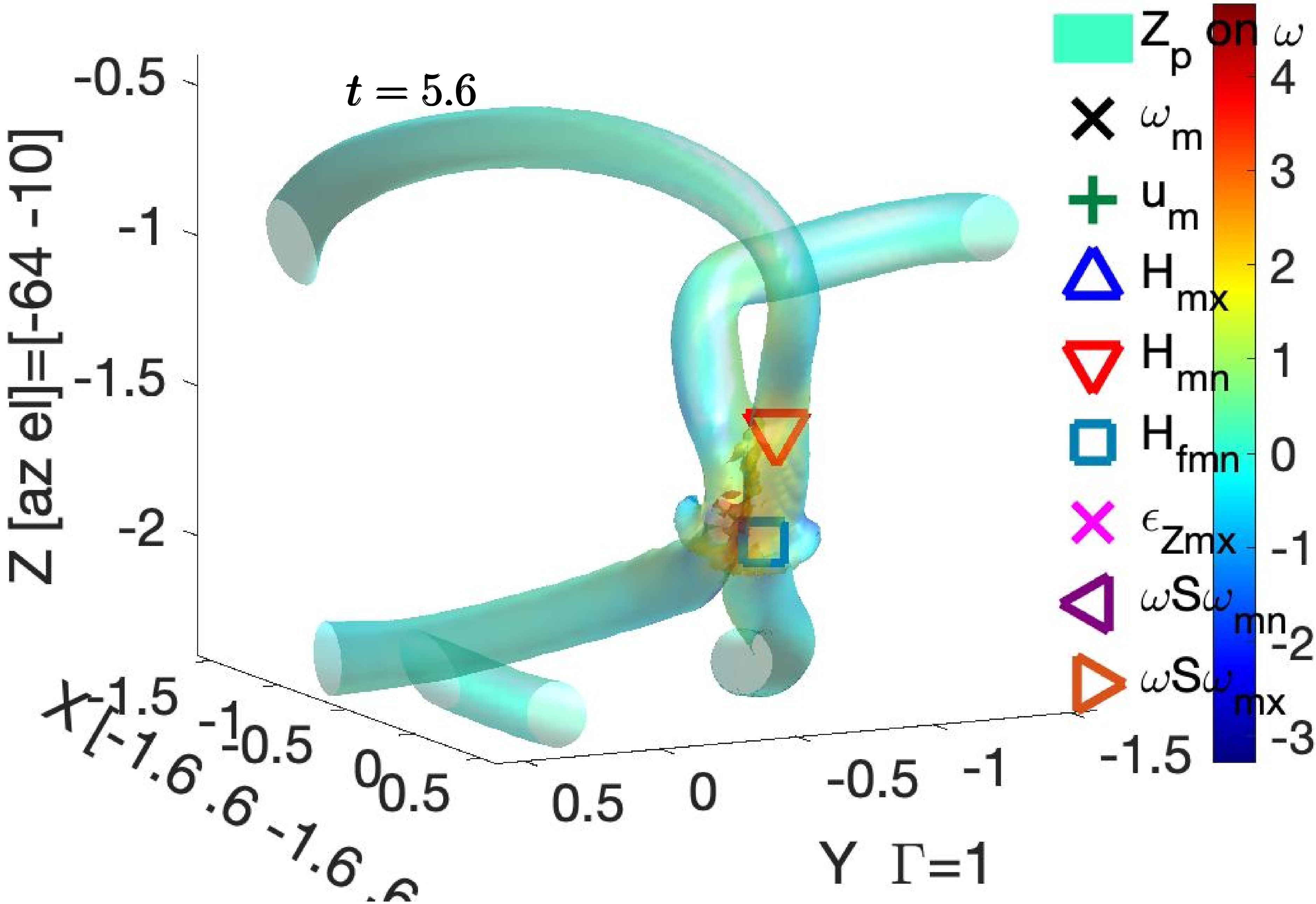
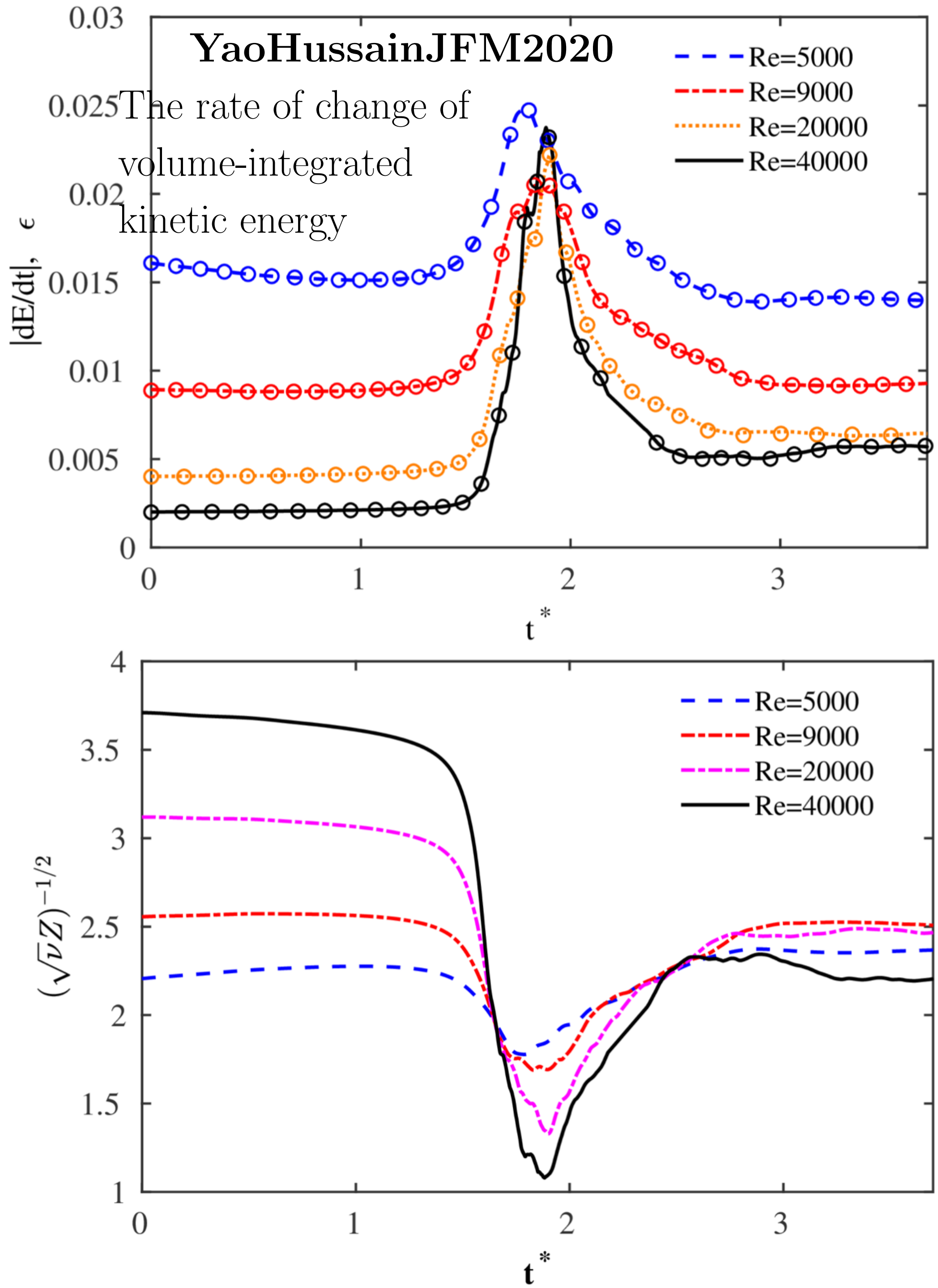
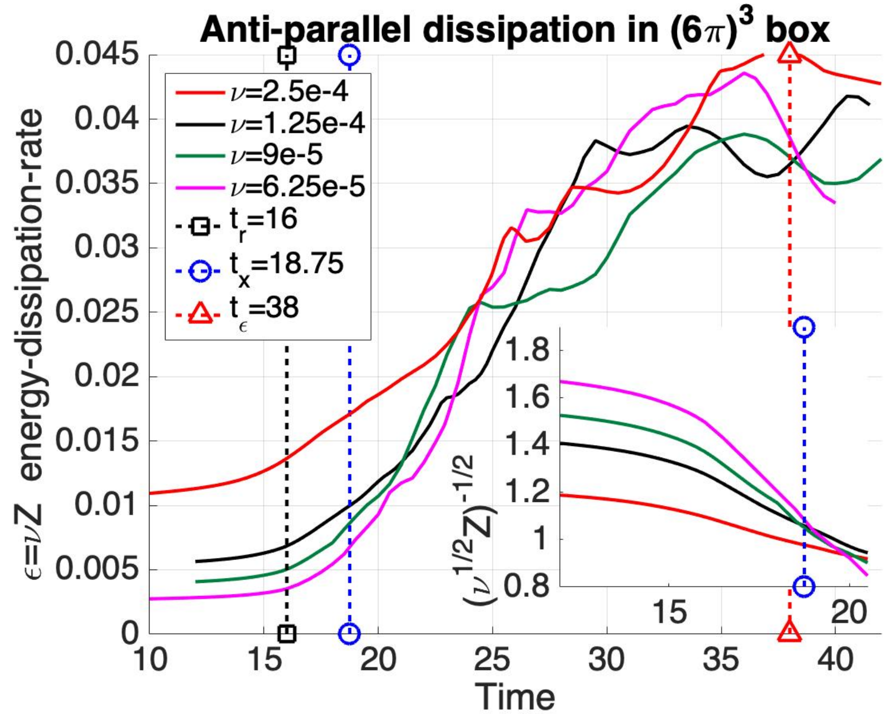
Y Γ=1
ω_m=36 |ω|=5.6
H_f on ω=5.6
X [az el]=[-42 58]



Negative helicity transport (cyan square) seems closely associated with **negative enstrophy production.** **What might this be telling us?** $H_f < 0$ where $Z_p < 0$? At $t = 0$ for three-fold symmetric $Z_p < 0$ dominates.



⇐ **Left:** From Yao/Hussain JFM (2020). Top: $|dE/dt|$ ($\circ$ symbols) and energy dissipation rate ϵ (lines) as function of $t = t/(2\pi b^2/\Gamma_0)$, b =initial separation. Converge and $k^{-5/3}$ spectra at $t^* = 1.9$. Bottom: $(\sqrt{\nu}Z)^{-1/2}$. Circulation-exchange and $\sqrt{\nu}Z$ scaling begin at $t^* = 1.5$ with convergence to $t_x = t^* = 1.8$.



⇑ **Above: Inset:** Initially similar: Reconnection (Γ-exchange and $\sqrt{\nu}Z$ scaling) begins later $t = 15$, completes by $t = 18$. **Full:** However convergence of ϵ and $k^{-5/3}$ are much later ($t_\epsilon \approx 2t_x$), persist and **not narrow**.

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