CONSTRUCTING WEAK SOLUTIONS - LESSONS FROM THE INVISCID BURGERS EQUATION

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ICTS INTERNATIONAL CENTRE for THEORETICAL SCIENCES

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INTRODUCTION

Understand the nature of **singularities** in ideal hydrodynamics, and the **weak solutions** that accompany them.

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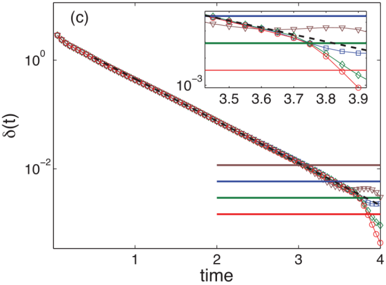
- Investigate a finite time **blow-up**.
- Tracing singularities ($z^* = \delta + \iota\mu$) by **Analyticity strip** method.
  - C. Sulem, P.L. Sulem, H.Frisch (1983).

$$|\tilde{u}(k,t)| \sim k^{-n} \exp\left[-\delta(t)k\right], \; k \gg 1 \tag{1}$$

  - **Loss of regularity** then would imply,

$$\lim_{t \to t^*} \delta(t) = 0 \tag{2}$$

- With smooth initial conditions, numerically solve the equations and extract $\delta(t)$ vs t.



Analyticity strip width for 3D Euler
- M.D. Bustamante & M.E. Brachet, 2012.

INTRODUCTION

Understand the nature of **singularities** in ideal hydrodynamics, and the **weak solutions** that accompany them.

Obstacle

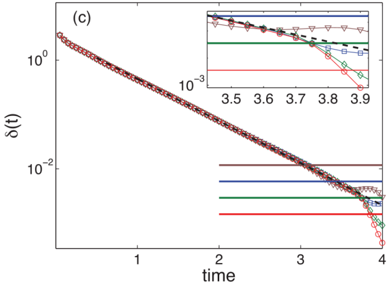
In a truncated system with finite degrees of freedom, as $\delta \sim k_{max}^{-1}$, small scale structures begin to thermalise implying,

$$E(k) \sim k^{d-1}$$

- Unreliable measurement of $\delta(t)$.
- 3D Euler - **Still a conjecture**

Goal

Providing a numerical prescription to prevent the truncated solutions from thermalising.



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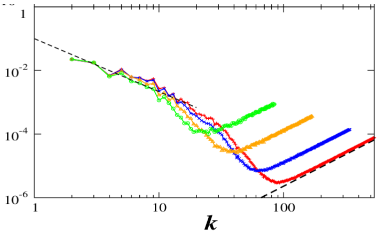


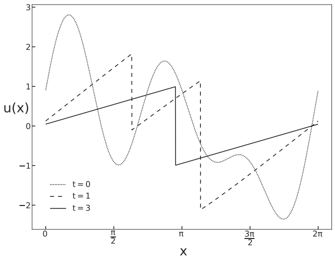
Thermalisation in truncated 3D Euler
- C.Cichowlas et al., 2005

INVISCID HYDRODYNAMICS - WEAK SOLUTIONS

1D Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

- Formation of shock in finite time.
  Singularity in $\frac{du}{dx}$.
  Burgers, 1974; Hopf, 1950; Cole, 1951; Fournier & U.Frisch, 1983.



Entropy solution

3D Incompressible Euler equation

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla P, \ \nabla \cdot u = 0$$

- Do the solutions blow up in finite time - **Unanswered**.
  L. Onsager, 1949: S. Orszag et al., 1983; J.T. Beale et al., 1984; M.D. Bustamante & M.E. Brachet, 2012.

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  Burgers, 1974; Hopf, 1950; Cole, 1951; Fournier & U.Frisch, 1983.
- Viscous Burgers equation shows anamalous dissipation as $\nu \to 0$.
- Entropy solution - dissipates weakly through these shocks - identical to the vanishing viscosity limit.
- Although truncated inviscid equation thermalises and differs radically from the **entropy solution.**
  S.S. Ray et al., 2011; P.C. Di Leoni et al., 2018

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- Navier-Stokes equation shows anamalous dissipation too.
- Numerical studies to trace the singularity, fails as the truncated Euler equations thermalise in finite time.
  C. Cichowlas et al., 2004,2005; W.J.T. Bos & J.P. Bertoglio, 2007

THERMALISATION IN INVISCID BURGERS EQUATION

Inviscid Burgers Equation

Numerical integration performed with pseudo spectral method on a Galerkin-truncated system

$$v(x) = \mathbb{P}_{k_G}[u(x)] = \sum_{|k|<k_G} \hat{u}_k e^{\iota k x}$$

$$\frac{\partial v}{\partial t} + \mathbb{P}_{k_G} \frac{1}{2} \frac{\partial v^2}{\partial x} = 0$$

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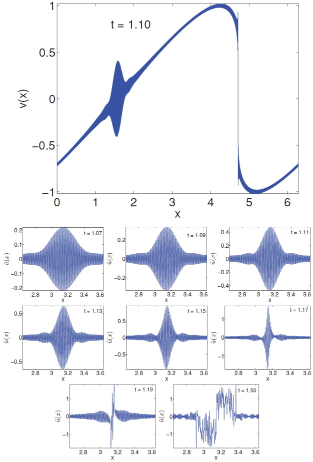
Finite number of modes, unable to develop small scale structures (beyond k_G). Resonates non-locally into what's termed as **Tygers** as $t \to t^*$.



(*top*) Birth of tyger, (*bottom*) showing discrepancy
-S.S. Ray et al., 2011

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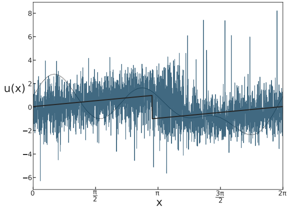
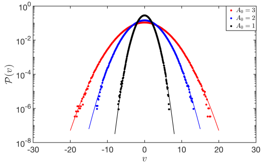
At $t \gg t^*$, fully thermalised solution with Gibbsian distribution, conserving energy. **No shocks, No dissipation**.



(dotted) Initial condition, *(blue)* truncated solution, *(black)* entropy solution
- S.D. Murugan et al., 2020.



PDF of thermalised velocity
- S.D. Murugan, *(unpub)*

NUMERICAL RECIPE TO SUPPRESS THERMALISATION

Tyger Purging - S.D.Murugan, U. Frisch, S. Nazarenko, N. Besse, and S.S. Ray, Phys. Rev. Research 2, 033202(2020)

In a truncated Burgers equation, implement a **selective removal of a narrow fourier space boundary layer Δk near k_G at discrete time intervals τ from the truncated solution.**

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- With the *ansatz*, parametrised by α, β

$$\boxed{\Delta k = k_G^{\beta},\ \tau = k_G^{-\alpha}} \tag{1}$$

- The numerical prescription to mildy modify the Galerkin-truncated equations to obtain solutions that do not thermalise,

$$\boxed{\frac{\partial w}{\partial t} + \mathbb{P}_{k_G} \frac{1}{2} \frac{\partial w^2}{\partial x} = 0} \tag{2}$$

$$\boxed{t_p = n\tau, n \in \mathbb{Z}; \quad \hat{w}_k :\equiv 0,\ \forall k_P \leqslant k \leqslant k_G} \tag{3}$$

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In a truncated Burgers equation, implement a **selective removal of a narrow fourier space boundary layer Δk near k_G at discrete time intervals τ from the truncated solution.**

- Intend to reset the energy cascade into the boundary layer after purging, mimicking a pre-shock.
- Temporal discreteness is key to counteract the truncation. Purging too soon or too late - effectiveness fades.

Converging to correct dissipation rate

In the study of Tygers (S.S. Ray et al., 2011), the envelope of oscillations in the discrepancy between entropy and truncated solution Fourier modes near k_G at the preshock time behaves as

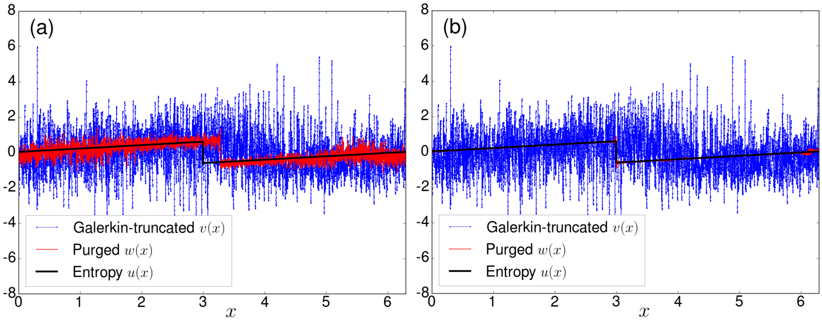
$$|\hat{v}_k| \sim \frac{1}{k_G} \exp\left[-\frac{c}{k_G^{1/3}}(k_G - k)\right]$$

Deducing a scaling for numerical dissipation by purging on k_G, leading to -

$$\beta \in [\frac{1}{3}, 1), \ \alpha + \beta \lessapprox 2$$

Necessary, Not sufficient. For e.g, $\alpha = 0.6$, $\beta = 0.4$ does not work, but $\alpha = \beta = 0.8$ work.

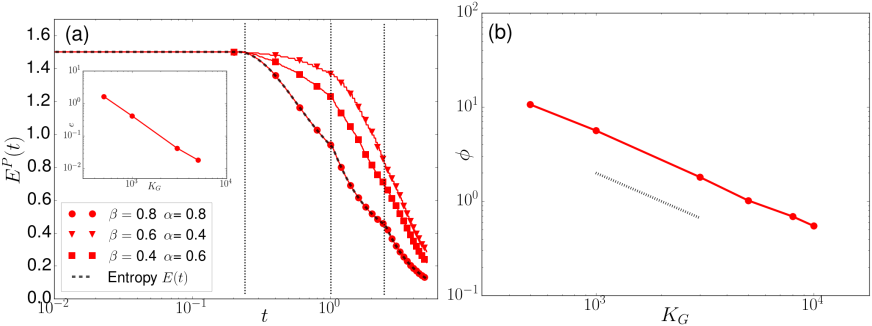
RESULTS - PURGED SOLUTIONS



(a) Unsuccessful purging $\alpha = 0.6$, $\beta = 0.4$, (b) Successful purging $\alpha = \beta = 0.8$ - S.D.Murugan, U. Frisch, S. Nazarenko, N. Besse, and S.S. Ray, Phys. Rev. Research 2, 033202 (2020)

Success of purging is irrespective of initial conditions, given $k_G \gg 1, \tau \gg \delta t$.

RESULTS - PURGED SOLUTIONS



(a) Energy decay of purged solution (*dashed showing for Entropy solution*) (b) Relative error - S.D.Murugan, U. Frisch, S. Nazarenko, N. Besse, and S.S. Ray, Phys. Rev. Research 2, 033202 (2020)

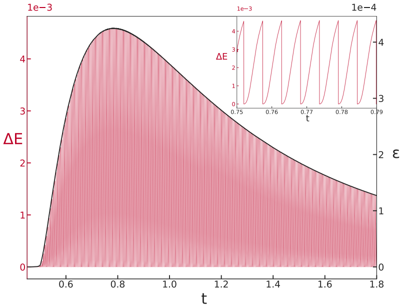
Purged solution picks-out correctly the shock strength, location, and velocity and merging of shocks.

RESULTS & CONCLUSION

The energy stored in those early tygers converges to the energy dissipated by the weak solution within the purging time, implying

$$\Delta E(t = n\tau) = \int_{t-\tau}^{t} \epsilon \, dt'$$

- The understanding of **tygers** as the source of thermalisation (S.S. Ray et al., 2011) is what allowed to devise this numerical recipe to suppress it.
- **Wavelet based filtering techniques** can also be used to suppress thermalisation and recover true solutions, but computationaly expensive compared to purging.
  - R.N. Van Yen et al., 2008; Pereira et al., 2013.



Energy in the boundary layer for the purged solution *(red)* and the extreme dissipation from the weak solution *(black)* - S.D. Murugan, *(Unpub.)*

CONCLUSION & FUTURE WORKS

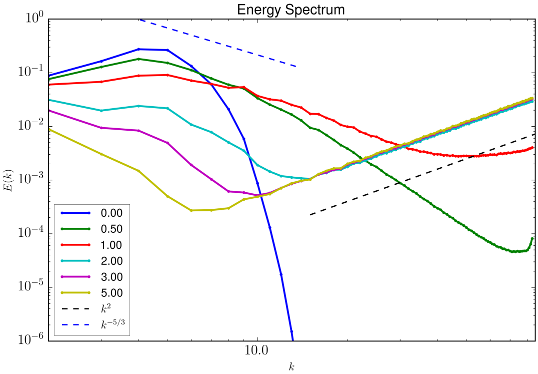
Weak solutions in 3D Euler

- To implement purging in truncated 3D Euler equations, we note
  - Analogous **3D Tygers** are not well known and understood.
  - Non-local interactions - **pressure**.
  - Navier-Stokes with Re $\rightarrow \infty$ - intermittent explained by **multifractality**. Burgers being bifractal.
- Nevertheless successful suppression of thermalisation implies,
  - Better numerical insights towards the celebrated **finite time blow-up** problem.
  - DNS with Re $= \infty$ with clear inertial range $E(k) \sim k^{-5/3}$ even for smaller resolutions with sub-grid scale modeling.
  - Singularity spectrum $f(\alpha)$ in the multifractal analysis for the obtained weak solutions.

THERMALISATION IN 3D EULER

3D Incompressible Euler equation

- In truncated 3D Euler, thermalisation first discovered by T.D. Lee, 1952, further developed by R.H.Kraichnan, 1967,1973.
- In recent years, (DNS at high resolution) studied by C. Cichowlas et al., 2005; M.E. Brachet et al., 2008,2009.
- All the studies were in spectral space.



THERMALISATION IN 3D EULER

Visualizing thermalisation in Real space

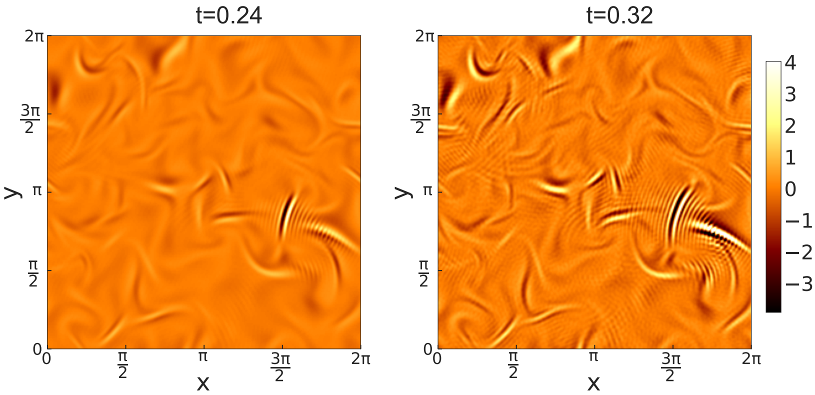
- Strain tensor is related to the vorticity field.

$$S_{ij} = \frac{3}{8\pi} \int dx' \, \left(\epsilon_{ikl} r_j + \epsilon_{jkl} r_i\right) \frac{r_k}{r^5} \omega_l(x')$$

- Decomposing the strain into local and background components, P.E. Hamlington et al., 2008.

$$S^{loc} \approx -\frac{R^2}{10} \nabla^2 S$$

THERMALISATION IN 3D EULER



Local strain components S_{xx} showing the onset *(left)*, growth*(right)* of thermalisation - S.D. Murugan *(Unpub.)*

Thank you,

Stay Home, Stay Safe.

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