Intermittency and thin sets in $3d$ Navier-Stokes Turbulence : A link with the Multifractal Model

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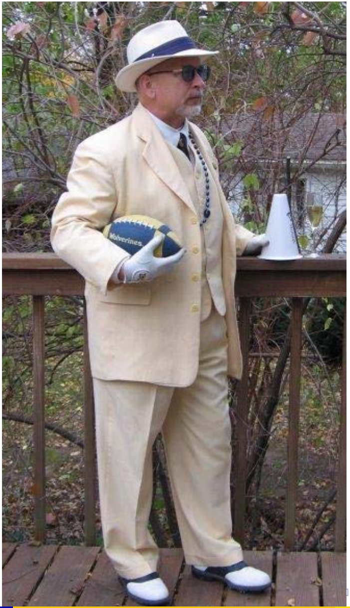
In memoriam : Charlie Doering (born 7th Jan 1956; died 15th May 2021)



Consider the 3D Navier-Stokes equations in the domain $[0, L]^3_{per}$

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f}(\boldsymbol{x}) \qquad \mathrm{div}\,\boldsymbol{u} = 0 \qquad (\mathrm{div}\,\boldsymbol{f} = 0)$$

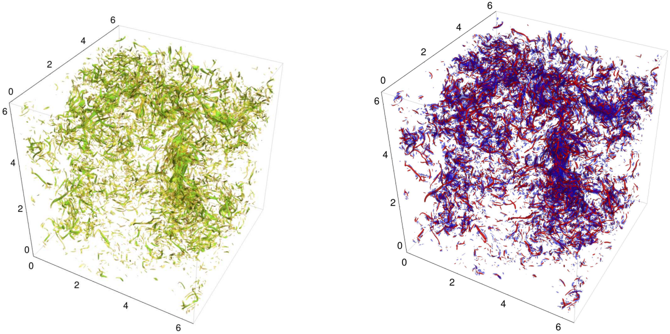


Figure: **Plots courtesy of J. R. Picardo and S. S. Ray.** **Left :** energy dissipation field $\varepsilon = 2\nu S_{i,j}S_{j,i}$ of a forced 512^3 NS flow at $Re_\lambda = 196$. **Right :** the field $Q = \frac{1}{2}\left(|\boldsymbol{\omega}|^2 - |S|^2\right)$.

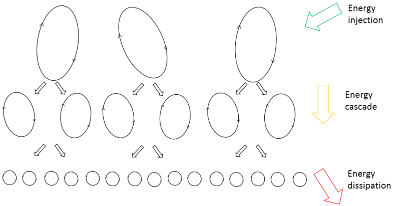
Question : Why does vorticity/strain accumulate on these 'thin sets'?

Some history of large-scale $3D$ NSE computations

Just for the record :

1. Orszag & Patterson 1972; Kerr 1985;
2. Eswaran & Pope 1988 ; Jimenez *et al* 1993 ;
3. Moin & Mahesh (Ann Rev FM 1998) ; Kurien & Taylor 2005 ;
4. Ishihara, Gotoh, Kaneda (Ann Rev FM 2009)
5. 4096^3 by Donzis, Yeung & Sreenivasan 2012 : $Re_\lambda \sim 1000$.
6. PK's talk at 14:20 gave an up-date on the current state of affairs : e.g. up to 18400^3 at ORNS. Also Dhawal Buaria's talk $12,288^3$, $Re_\lambda \sim 1300$.
7. (i) 8000^3 computation – Ishihara, Elsinga & Hunt (PrRS 2020).
(ii) Elsinga, Ishihara, Goudar, da Silva & Hunt (2017).
(iii) Hunt, Ishihara, Worth & Kaneda (2013, 2017).

Turbulent cascades & higher derivatives



Numerical simulations of the $3D$ Navier-Stokes equations show that finer and finer vortical structures appear as resolution increases involving inverse scales much smaller than λ_k.

Define a doubly-labeled set of volume integrals for $1 \leq n < \infty\,;\, 1 \leq m \leq \infty$ in d-dimensions

$$H_{n,m,d} = \int_{V_d} |\nabla^n \boldsymbol{u}|^{2m} dV_d$$

In dimensionless form :

$$F_{n,m,d} = \nu^{-1} L^{1/\alpha_{n,m,d}} H_{n,m,d}^{1/2m}\,, \qquad \alpha_{n,m,d} = \frac{2m}{2m(n+1)-d}\,,$$

1. Derivatives are sensitive to ever finer length scales in the flow.
2. Higher values of m pick out the larger spikes, with the $m = \infty$ case representing the maximum norm.

The NSEs have the scale invariance :

$$\boldsymbol{u}(\boldsymbol{x},\, t) \to \lambda^{-1}\boldsymbol{u}\left(\boldsymbol{x}/\lambda,\, t/\lambda^2\right) \qquad \Rightarrow \qquad F_{n,m,d} \to F_{n,m,d}\,.$$

In the following $\langle\cdot\rangle_T$ means time average up to time T :

Result

On periodic BCs with $n \geq 1$ & $1 \leq m \leq \infty$, d-dim NS-weak solutions obey

$$\left\langle F_{n,m,d}^{(4-d)\alpha_{n,m,d}} \right\rangle_T \leq c_{n,m,d}\, Re^3\,, \qquad \textit{for } d = 2,\, 3$$

For $d = 1$ the same result holds for Burgers' equation. JDG : EPL 2020.

- For $d = 3$ when $n = 1$, $m = 1$ gives the standard $\varepsilon \leq L^{-4}\nu^3 Re^3$ from which the Kolmogorov length λ_k is estimated

$$\lambda_k^{-1} = \left(\frac{\varepsilon}{\nu^3}\right)^{1/4} \qquad \Rightarrow \qquad L\lambda_k^{-1} \leq Re^{3/4}\,.$$

- Is there a continuum of length scales corresponding to n, $m > 1$?

Define a set of t-dependent length-scales $\{\lambda_{n,m,d}(t)\}$ s.t.

$$\left(\frac{L}{\lambda_{n,m,d}}\right)^{-d} H_{n,m,d} = \lambda_{n,m,d}^{-2m(n+1)+d}\nu^{2m}$$

from which we discover

$$\left(L\lambda_{n,m,d}^{-1}\right)^{n+1} = F_{n,m,d} \qquad \text{with} \qquad \alpha_{n,m,d} = \frac{2m}{2m(n+1)-d}$$

For NS weak solutions, when $n \geq 1$ and $1 \leq m \leq \infty$

$$\left\langle L\lambda_{n,m,d}^{-1}\right\rangle_T \leq c_{n,m,d} Re^{\frac{3}{(4-d)(n+1)\alpha_{n,m,d}}} + O\left(T^{-1}\right).$$

The upper bound has a finite limit : Richardson and Kolmogorov were correct!

$$\lim_{\mathbf{n,m}\to\infty} \frac{\mathbf{3}}{(\mathbf{4}-\mathbf{d})(\mathbf{n}+\mathbf{1})\alpha_{\mathbf{n,m,d}}} = \frac{\mathbf{3}}{\mathbf{4}-\mathbf{d}}$$

More on scaling in d dimensions

Examine the exponent of $F_{n,m,d}$: one finds that

$$(4-d)\alpha_{n,m,d} = \frac{2m(4-d)}{2m(n+1)-d} \quad \text{increases} \qquad \text{as} \qquad d \searrow 0\,.$$

- **Thus as the dimension decreases the dissipation increases which implies more, not less, regularity.**
- *Numerical simulations suggest that a flow may adjust itself to find the smoothest, most dissipative set on which to operate.*
- This runs counter to a commonly held theory of viscous turbulence in which singularities have been long-standing candidates as the underlying cause of turbulent dynamics.
- (i) JDG : J. Nonlin. Sci., **29**(1), 215228, 2019
  (ii) JDG : *Turbulent cascades & thin sets in* $3D$ *NS-turbulence* EPL 2020

The Multifractal Model (MFM) of Parisi and Frisch : I

The p-th order velocity structure function S_p should scale as

$$S_p(r) = \left\langle |\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})|^p \right\rangle_{st.av.} \sim r^{hp}\,.$$

- K41 theory says that $h = \frac{1}{3}$ to ensure that the energy dissipation rate ε is homogeneous in space and time. Thus $S_p \sim r^{p/3}$. When $p = 3$ the right hand side is equal to $-\frac{4}{5}\varepsilon r$ which is Kolmogorov's four-fifths law.
- Parisi and Frisch (1985) then relaxed the enforcement of $h = \frac{1}{3}$ to allow a continuous spectrum of exponents h, provided the dissipation rate ε is constant "on the average".
- In the MFM's original formulation $P_r(h)$, the probability of observing a given scaling exponent h at the scale r was computed by assuming that each value of h belongs to a given fractal set of dimension $D(h)$. A more precise mathematical definition can be established by using Large Deviation Theory where $P_r(h)$ is chosen as (see Eyink 2008)

$$P_r(h) \sim r^{C(h)}\,.$$

- $C(h)$ is the multi-fractal spectrum. It has encoded within it all the properties of flow intermittency. One can write $d = D(h) + C(h)$.

The structure functions $S_p(r)$, instead of taking their K41-form with $h = \frac{1}{3}$, are now expressed as

$$S_p(r) \sim r^{\zeta_p} , \qquad \zeta_p = \inf_h \left[hp + C(h) \right] .$$

A classic sign of intermittency is that ζ_p is a *concave curve below linear*. In the $3d$ computations below, note that $\zeta_3 = 1$:

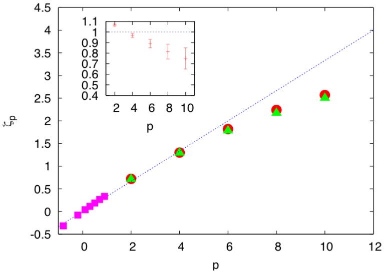


Figure: Taken from http://www.scholarpedia.org/article/Turbulence curated by Uriel Frisch. The value of the exponents obtained by two independent direct numerical simulations of homogeneous isotropic turbulence at very high resolution.

Paladin and Vulpiani (1987) suggested an h-dependent dissipation scale $L\eta_h^{-1} \sim Re^{\frac{1}{1+h}}$. We use the scaling $\eta_h \sim \nu^{\frac{1}{1+h}}$ to obtain the correspondence

$$L^{-3}\int_{\mathcal{V}_\lceil} |\nabla^n \boldsymbol{u}|^{2m}\, dV_d \quad \longleftrightarrow \quad \int_h \eta_h^{2m(h-n)} P_{\eta_h}(h)dh\,,$$

Apply this to our estimate for $\left\langle F_{n,m,d}^{(4-d)\alpha_{n,m,d}} \right\rangle_T$, **for all derivatives** :

- $\mathbf{h} \geq (\mathbf{1}-\mathbf{d})/\mathbf{3}$; for $d = 3$ we have $\mathbf{h} \geq -\mathbf{2}/\mathbf{3}$.
- $\mathbf{C(h)} \geq \mathbf{1}-\mathbf{3h}$: **consistent with the four-fifths law.** Also $C(h_{min}) \geq d$.

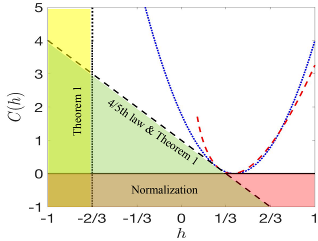


Figure: The admissibility range of $C(h)$ when $d = 3$ including $C(h) \geq 1-3h$. The blue dotted line : log-normal model with $b = 0.045$; red dashed line : log-Poisson model with $\beta = 2/3$.

Avoidance of the $3d$ NS singular set?

In $d = 3$ dimensions, the range of h is now

$$-2/3 \leq h \leq 1/3$$

thus implying a wide range of fractal dimensions.

1. Caffarelli, Kohn and Nirenberg (1982) developed the idea of suitable weak solutions of the $3d$ NSEs. The singular set in space-time has zero one-dimensional Hausdorff measure.
2. Their result shows that in the limit $r \to 0$, as solutions approach the CKN singular set, the velocity field $\boldsymbol{u}$ must obey
$$|\boldsymbol{u}| > \frac{const}{r}, \quad \text{as} \quad r \to 0\,.$$
The r^{-1} lower bound suggests a minimal rate of approach to the the CKN singular set. **The corresponding value of h is $h = -1$.**
3. Thus, our lower bound $h \geq -2/3$ keeps solutions away from the singular set.

Dubrulle and Gibbon : arXiv:2102.00189v3 [physics.flu-dyn]